

THE REPRESENTATIONS OF CYCLOTOMIC BMW ALGEBRAS, II

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ABSTRACT. In this paper, we go on Rui-Xu's work on cyclotomic Birman-Wenzl algebras $\mathcal{B}_{r,n}$ in [19]. In particular, we use the representation theory of cellular algebras in [11] to classify the irreducible $\mathcal{B}_{r,n}$ -modules for all positive integers r and n . By constructing cell filtrations for all cell modules of $\mathcal{B}_{r,n}$, we compute the discriminants associated to all cell modules for $\mathcal{B}_{r,n}$. Via such discriminants together with induction and restriction functors given in section 5, we determine explicitly when $\mathcal{B}_{r,n}$ is semisimple over a field. This generalizes our previous result on Birman-Wenzl algebras in [17].

1. INTRODUCTION

Let $\mathcal{B}_{r,n}$ be the cyclotomic Birman-Wenzl algebras defined in [12]. Motivated by Ariki, Mathas and Rui's work on cyclotomic Nazarov-Wenzl algebras [4], Rui and Xu [19] proved that $\mathcal{B}_{r,n}$ is cellular over R for all positive odd integers r under the so-called \mathbf{u} -admissible conditions (see the assumption 2.2). Moreover, they have classified the irreducible $\mathcal{B}_{r,n}$ -modules.

In this paper, we will prove that $\mathcal{B}_{r,n}$ is cellular over R for all positive integers r under the \mathbf{u} -admissible conditions. By using arguments in [19], we classify the irreducible $\mathcal{B}_{r,n}$ -modules over an arbitrary field. This completes the classification of irreducible $\mathcal{B}_{r,n}$ -modules over a field. We remark that Yu [20] first proved that $\mathcal{B}_{r,n}$ is cellular over R under the similar conditions. However, she did assume that the parameter ω_0 , which is given in Definition 2.1, is invertible when she proved that $\mathcal{B}_{r,n}$ is cellular.

Given a cell module M of $\mathcal{B}_{r,n}$. Following [17], we construct a $\mathcal{B}_{r,n-1}$ -filtration for M . Via it, we construct an R -basis for M , called JM-basis in the sense of [15]. This enables us to use standard arguments in [15] to construct an orthogonal basis for M under so called **separate condition** in the sense of [15]. The key is that the Gram determinants associated to M which are defined by the JM-basis and the previous orthogonal basis are the same. We will give a recursive formula to compute the later determinant.

Motivated by [9], we construct restriction functor \mathcal{F} and induction functor \mathcal{G} which set up a relationship between the category of $\mathcal{B}_{r,n}$ -modules and the category of $\mathcal{B}_{r,n-2}$ -modules. Via \mathcal{F} and \mathcal{G} together with certain explicit formulae on Gram determinants, we determine explicitly when $\mathcal{B}_{r,n}$ is semisimple over a field.

We organize the paper as follows. In Section 2, we prove that $\mathcal{B}_{r,n}$ is cellular over R for all positive integers r and n . We also classify the irreducible $\mathcal{B}_{r,n}$ -modules. In section 3, we construct the JM-basis and an orthogonal basis for each cell module of $\mathcal{B}_{r,n}$. In section 4, we compute the discriminants associated to all cell modules of $\mathcal{B}_{r,n}$. Restriction functor \mathcal{F} and induction functor \mathcal{G} will be constructed in section 5. In section 6, we determine explicitly when $\mathcal{B}_{r,n}$ is semisimple over an arbitrary field.

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2. THE CYCLOTOMIC BIRMAN-WENZL ALGEBRAS

Throughout the paper, we fix two positive integers r and n . Let R be a commutative ring which contains the identity 1 and invertible elements $q^{\pm 1}, u_1^{\pm 1}, \dots, u_r^{\pm 1}, \varrho^{\pm 1}, \delta^{\pm 1}$ such that $\delta = q - q^{-1}$ and $\omega_0 = 1 - \delta^{-1}(\varrho - \varrho^{-1})$.

Definition 2.1. [12] The cyclotomic Birman-Wenzl algebra $\mathcal{B}_{r,n}$ is the unital associative R -algebra generated by $\{T_i, E_i, X_j^{\pm 1} \mid 1 \leq i < n \text{ and } 1 \leq j \leq n\}$ subject to the following relations:

- a) $X_i X_i^{-1} = X_i^{-1} X_i = 1$ for $1 \leq i \leq n$.
- b) (Kauffman skein relation) $1 = T_i^2 - \delta T_i + \delta \varrho E_i$, for $1 \leq i < n$.
- c) (braid relations)
 - (i) $T_i T_j = T_j T_i$ if $|i - j| > 1$,
 - (ii) $T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1}$, for $1 \leq i < n - 1$,
 - (iii) $T_i X_j = X_j T_i$ if $j \neq i, i + 1$.
- d) (Idempotent relations) $E_i^2 = \omega_0 E_i$, for $1 \leq i < n$.
- e) (Commutation relations) $X_i X_j = X_j X_i$, for $1 \leq i, j \leq n$.
- f) (Skein relations)
 - (i) $T_i X_i - X_{i+1} T_i = \delta X_{i+1} (E_i - 1)$, for $1 \leq i < n$,
 - (ii) $X_i T_i - T_i X_{i+1} = \delta (E_i - 1) X_{i+1}$, for $1 \leq i < n$.
- g) (Unwrapping relations) $E_1 X_1^a E_1 = \omega_a E_1$, for $a \in \mathbb{Z}$.
- h) (Tangle relations)
 - (i) $E_i T_i = \varrho E_i = T_i E_i$, for $1 \leq i \leq n - 1$,
 - (ii) $E_{i+1} E_i = E_{i+1} T_i T_{i+1} = T_i T_{i+1} E_i$, for $1 \leq i \leq n - 2$.
- i) (Untwisting relations)
 - (i) $E_{i+1} E_i E_{i+1} = E_{i+1}$ for $1 \leq i \leq n - 2$,
 - (ii) $E_i E_{i+1} E_i = E_i$, for $1 \leq i \leq n - 2$.
- j) (Anti-symmetry relations) $E_i X_i X_{i+1} = E_i = X_i X_{i+1} E_i$, for $1 \leq i < n$.
- k) (Cyclotomic relation) $(X_1 - u_1)(X_1 - u_2) \cdots (X_1 - u_r) = 0$

For each $x \in R$, let

$$\gamma_r(x) = \begin{cases} 1, & \text{if } 2 \nmid r, \\ -x, & \text{if } 2 \mid r. \end{cases}$$

In the remainder of this paper, We use \mathbf{u} (resp. Ω) to denote (u_1, u_2, \dots, u_r) (resp. $\{\omega_a \mid a \in \mathbb{Z}\}$). In order to show that $\mathcal{B}_{r,n}$ is free over R , Rui and Xu introduced the \mathbf{u} -admissible conditions in [19, 3.15] as follows.

Assumption 2.2. $\Omega \cup \{\varrho\}$ is called \mathbf{u} -admissible if

$$\varrho^{-1} = \alpha \prod_{\ell=1}^r u_\ell, \text{ and } \omega_a = \sum_{j=1}^r u_j^a \gamma_j, \forall a \in \mathbb{Z}$$

where

- (1) $\gamma_i = (\gamma_r(u_i) + \delta^{-1} \varrho (u_i^2 - 1) \prod_{j \neq i} u_j) \prod_{j \neq i} \frac{u_i u_j - 1}{u_i - u_j}$,
- (2) $\alpha \in \{1, -1\}$ if $2 \nmid r$ and $\alpha \in \{q^{-1}, -q\}$, otherwise.
- (3) $\omega_0 = \delta^{-1} \varrho (\prod_{\ell=1}^r u_\ell^2 - 1) + 1 - \frac{(-1)^{r+1}}{2} \alpha^{-1} \varrho^{-1}$.

Note that there are infinite equalities in the definition of \mathbf{u} -admissible conditions in Assumption 2.2. It has been proved in [19, 3.17] that $\omega_j, \forall j \in \mathbb{Z}$, are determined by $\omega_i, 0 \leq i \leq r - 1$. Furthermore, all ω_i are elements in $\mathbb{Z}[u_1^{\pm 1}, \dots, u_r^{\pm 1}, q^{\pm 1}, \delta^{-1}]$ [19, 3.11]. Therefore, $\omega_i \in R$ for all $i \in \mathbb{Z}$ if they are given in the Assumption 2.2.

In the remainder of this paper, unless otherwise stated, we always keep the Assumption 2.2 when we discuss $\mathcal{B}_{r,n}$ over R .

It has been proved in [19] that $\mathcal{B}_{r,n}$ is a free R -module with rank $r^n(2f-1)!!$ when r is odd. We will prove that $\mathcal{B}_{r,n}$ is cellular over R with rank $r^n(2f-1)!!$ when r is even. We start by recalling the definition of Ariki-Koike algebras in [2].

The Ariki-Koike algebra [2] $\mathcal{H}_{r,n}(\mathbf{u}) := \mathcal{H}_{r,n}$ is the unital associative R -algebra generated by y_1, \dots, y_n and g_1, g_2, \dots, g_{n-1} subject to the following relations:

- a) $(g_i - q)(g_i + q^{-1}) = 0$, if $1 \leq i \leq n-1$,
- b) $g_i g_j = g_j g_i$, if $|i - j| > 1$,
- c) $g_i g_{i+1} g_i = g_{i+1} g_i g_{i+1}$, for $1 \leq i < n-1$,
- d) $g_i y_j = y_j g_i$, if $j \neq i, i+1$,
- e) $y_i y_j = y_j y_i$, for $1 \leq i, j \leq n$,
- f) $y_{i+1} = g_i y_i g_i$, for $1 \leq i \leq n-1$,
- g) $(y_1 - u_1)(y_1 - u_2) \dots (y_1 - u_r) = 0$.

Let $\mathcal{E}_n = \mathcal{B}_{r,n} E_1 \mathcal{B}_{r,n}$ be the two-sided ideal of $\mathcal{B}_{r,n}$ generated by E_1 . It is proved in [19, 5.2] that $\mathcal{H}_{r,n} \cong \mathcal{B}_{r,n} / \mathcal{E}_n$. The corresponding R -algebraic isomorphism is determined by

$$\varepsilon_n : g_i \mapsto T_i + \mathcal{E}_n, \text{ and } y_j \mapsto X_j + \mathcal{E}_n,$$

for $1 \leq i < n$ and $1 \leq j \leq n$.

Let \mathfrak{S}_n be the symmetric group on $\{1, 2, \dots, n\}$. Then \mathfrak{S}_n is generated by $s_i = (i, i+1)$, $1 \leq i \leq n-1$. If $w = s_{i_1} \dots s_{i_k} \in \mathfrak{S}_n$ is a reduced expression of w , then we write $T_w = T_{i_1} T_{i_2} \dots T_{i_k} \in \mathcal{B}_{r,n}$. It has been pointed out in [19] that T_w is independent of a reduced expression of w . We denote by

$$(2.3) \quad \mathbb{N}_r = \left\{ i \in \mathbb{Z} \mid -\lfloor \frac{r}{2} \rfloor + \frac{1}{2}(1 + (-1)^r) \leq i \leq \lfloor \frac{r}{2} \rfloor \right\}.$$

Given a non-negative integer f with $f \leq \lfloor n/2 \rfloor$. Following [19, 5.5], we define

$$(2.4) \quad \mathcal{D}_{f,n} = \left\{ s_{n-2f+1, i_f} s_{n-2f+2, j_f} \dots s_{n-1, i_1} s_{n, j_1} \mid \begin{array}{l} 1 \leq i_f < \dots < i_1 \leq n, \\ 1 \leq i_k < j_k \leq n-2k+2, 1 \leq k \leq f \end{array} \right\},$$

where

$$s_{i,j} = \begin{cases} s_{i-1} s_{i-2} \dots s_j, & \text{if } i > j, \\ s_i s_{i+1} \dots s_{j-1}, & \text{if } i < j, \\ 1, & \text{if } i = j. \end{cases}$$

Let $\mathfrak{B}_f \subset \mathfrak{S}_n$ be the subgroup generated by $s_{n-2i+2} s_{n-2i+1} s_{n-2i+3} s_{n-2i+2}$, $2 \leq i \leq f$, and s_{n-1} . Then $\mathcal{D}_{f,n}$ is a right coset representatives for $\mathfrak{S}_{n-2f} \times \mathfrak{B}_f$ in \mathfrak{S}_n (see e.g. [19]).

For each $d = s_{n-2f+1, i_f} s_{n-2f+2, j_f} \dots s_{n-1, i_1} s_{n, j_1} \in \mathcal{D}_{f,n}$, let κ_d be the n -tuple (k_1, \dots, k_n) such that $k_i \in \mathbb{N}_r$ and $k_i \neq 0$ only for $i = i_1, i_2, \dots, i_f$. Note that κ_d may be equal to κ_e although $e \neq d$ for $e, d \in \mathcal{D}_{f,n}$. We set $X^{\kappa_d} = \prod_{i=1}^n X_i^{\kappa_{i,d}}$. By Definition 2.1,

$$(2.5) \quad T_d X^{\kappa_d} = T_{n-2f+1, i_f} X_{i_f}^{\kappa_{i_f}} T_{n-2f+2, j_f} \dots T_{n-1, i_1} X_{i_1}^{\kappa_{i_1}} T_{n, j_1},$$

where $T_{i,j} = T_{s_{i,j}}$. For convenience, let

$$(2.6) \quad \mathbb{N}_r^{f,n} = \{\kappa_d \mid d \in \mathcal{D}_{f,n}\}.$$

Recall that a **composition** λ of m is a sequence of non-negative integers $(\lambda_1, \lambda_2, \dots)$ such that $|\lambda| := \lambda_1 + \lambda_2 + \dots = m$. λ is called a **partition** if $\lambda_i \geq \lambda_{i+1}$ for all positive integers i . Similarly, an r -**partition** (resp. r -**composition**) of m is an ordered r -tuple $\lambda = (\lambda^{(1)}, \dots, \lambda^{(r)})$ of partitions (resp. compositions) $\lambda^{(s)}$, $1 \leq s \leq r$, such that $|\lambda| := |\lambda^{(1)}| + \dots + |\lambda^{(r)}| = m$. In the remainder of this paper, we use multipartitions (resp. multicompositions) instead of r -partitions (resp. r -compositions). Let $\Lambda_r^+(m)$ (resp. $\Lambda_r(m)$) be the set of all multipartitions (resp. multicompositions) of m .

It is known that both $\Lambda_r^+(m)$ and $\Lambda_r(m)$ are posets with the dominance order \supseteq defined on them. We have $\lambda \leq \mu$ if

$$\sum_{j=1}^{i-1} |\lambda^{(j)}| + \sum_{k=1}^l \lambda_k^{(i)} \leq \sum_{j=1}^{i-1} |\mu^{(j)}| + \sum_{k=1}^l \mu_k^{(i)}$$

for $1 \leq i \leq r$ and $l \geq 0$. We write $\lambda \triangleleft \mu$ if $\lambda \leq \mu$ and $\lambda \neq \mu$. Let

$$\Lambda_{r,n}^+ = \{(k, \lambda) \mid 0 \leq k \leq \lfloor n/2 \rfloor, \lambda \in \Lambda_r^+(n-2k)\}.$$

Then $\Lambda_{r,n}^+$ is a poset with \supseteq as the partial order on it. More explicitly, $(k, \lambda) \supseteq (\ell, \mu)$ for $(k, \lambda), (\ell, \mu) \in \Lambda_{r,n}^+$ if either $k > \ell$ in the usual sense or $k = \ell$ and $\lambda \supseteq \mu$. Here \supseteq is the dominance order defined on $\Lambda_r^+(n-2k)$.

The Young diagram $Y(\lambda)$ of a partition $\lambda = (\lambda_1, \lambda_2, \dots)$ is a collection of boxes arranged in left-justified rows with λ_i boxes in the i -th row of $Y(\lambda)$. A λ -tableau \mathbf{t} is obtained from $Y(\lambda)$ by inserting $\{1, \dots, n\}$ into each box of $Y(\lambda)$ without repetition. If the entries in \mathbf{t} increase from left to right in each row and from top to bottom in each column, then \mathbf{t} is called a standard λ -tableau.

If $\lambda = (\lambda^{(1)}, \dots, \lambda^{(r)}) \in \Lambda_r^+(n)$, then the Young diagram $Y(\lambda)$ is an ordered Young diagrams $(Y(\lambda^{(1)}), \dots, Y(\lambda^{(r)}))$. In this case, a λ -tableau \mathbf{t} is $(\mathbf{t}_1, \dots, \mathbf{t}_r)$ where each $\mathbf{t}_i, 1 \leq i \leq r$ is a $\lambda^{(i)}$ -tableau. If the entries in each \mathbf{t}_i increase from left to right in each row and from top to bottom in each column, then \mathbf{t} is called standard. Let $\mathcal{T}^{std}(\lambda)$ be the set of all standard λ -tableaux.

Suppose $\lambda \in \Lambda_r^+(n)$. It is well-known that $\mathcal{T}^{std}(\lambda)$ is a poset with dominance order \supseteq on it. For each $\mathbf{s} \in \mathcal{T}^{std}(\lambda)$ and a positive integer $i \leq n$, let $\mathbf{s} \downarrow_i$ be obtained from \mathbf{s} by deleting all entries in \mathbf{s} greater than i . Let \mathbf{s}_i be the multipartition of i such that $\mathbf{s} \downarrow_i$ is the \mathbf{s}_i -tableau. Then $\mathbf{s} \supseteq \mathbf{t}$ if and only if $\mathbf{s}_i \supseteq \mathbf{t}_i$ for all $i, 1 \leq i \leq n$. Write $\mathbf{s} \triangleright \mathbf{t}$ if $\mathbf{s} \supseteq \mathbf{t}$ and $\mathbf{s} \neq \mathbf{t}$.

It is well-known that \mathfrak{S}_n acts on a λ -tableau by permuting its entries. Let \mathbf{t}^λ be the λ -tableau obtained from $Y(\lambda)$ by adding $1, 2, \dots, n$ from left to right along the rows of $Y(\lambda^{(1)})$, $Y(\lambda^{(2)})$, etc. For example, if $\lambda = ((3, 2), (2, 1), (1, 1)) \in \Lambda_3^+(10)$, then

$$\mathbf{t}^\lambda = \left(\begin{array}{|c|c|c|} \hline 1 & 2 & 3 \\ \hline 4 & 5 & \\ \hline \end{array}, \begin{array}{|c|c|} \hline 6 & 7 \\ \hline 8 & \\ \hline \end{array}, \begin{array}{|c|} \hline 9 \\ \hline 10 \\ \hline \end{array} \right)$$

Let \mathfrak{S}_λ be the Young subgroup associated to the multipartition λ . Then \mathfrak{S}_λ is the row stabilizer of \mathbf{t}^λ . Let $a_i = \sum_{j=1}^i |\lambda^{(j)}|$, $1 \leq i \leq r$ and $a_0 = 0$. For each λ -tableau \mathbf{t} , there is a unique element, say $d(\mathbf{t}) \in \mathfrak{S}_n$, such that $\mathbf{t} = \mathbf{t}^\lambda d(\mathbf{t})$. Suppose that $\mathbf{s}, \mathbf{t} \in \mathcal{T}^{std}(\lambda)$ where $\lambda \in \Lambda_r^+(n-2f)$ for some non-negative integer $f \leq \lfloor \frac{n}{2} \rfloor$. It is defined in [19, 5.7] that

$$(2.7) \quad M_{\mathbf{st}} = T_{d(\mathbf{s})}^* \cdot \prod_{s=2}^r \prod_{i=1}^{a_{s-1}} (X_i - u_s) \sum_{w \in \mathfrak{S}_\lambda} q^{l(w)} T_w \cdot T_{d(\mathbf{t})},$$

where $*$ is the R -linear anti-involution on $\mathcal{B}_{r,n}$, which fixes T_i and X_j , $1 \leq i \leq n-1$ and $1 \leq j \leq n$. Note that

$$(2.8) \quad \mathbf{m}_{\mathbf{st}} := \varepsilon_{n-2f}^{-1} (M_{\mathbf{st}} + \mathcal{E}_n)$$

is the Murphy basis element for Ariki-Koike algebra $\mathcal{H}_{r,n-2f}$ in [7].

We define $M_\lambda = M_{\mathbf{t}^\lambda \mathbf{t}^\lambda}$ and $E^{f,n} = E_{n-1} E_{n-3} \cdots E_{n-2f+1}$ and $\mathcal{B}_{r,n}^f = \mathcal{B}_{r,n} E^{f,n} \mathcal{B}_{r,n}$ for each non-negative integer $f \leq \lfloor n/2 \rfloor$. Therefore, there is a filtration of two-sided ideals of $\mathcal{B}_{r,n}$ as follows:

$$(2.9) \quad \mathcal{B}_{r,n} = \mathcal{B}_{r,n}^0 \supset \mathcal{B}_{r,n}^1 \supset \cdots \supset \mathcal{B}_{r,n}^{\lfloor \frac{n}{2} \rfloor} \supset 0.$$

Definition 2.10. Suppose that $0 \leq f \leq \lfloor \frac{n}{2} \rfloor$ and $\lambda \in \Lambda_r^+(n-2f)$. Define $\mathcal{B}_{r,n}^{\triangleright(f,\lambda)}$ to be the two-sided ideal of $\mathcal{B}_{r,n}$ generated by $\mathcal{B}_{r,n}^{f+1}$ and S where

$$S = \{ E^{f,n} M_{\mathbf{st}} \mid \mathbf{s}, \mathbf{t} \in \mathcal{T}^{std}(\mu) \text{ and } \mu \in \Lambda_r^+(n-2f) \text{ with } \mu \triangleright \lambda \}.$$

We also define $\mathcal{B}_{r,n}^{\triangleright(f,\lambda)} = \sum_{\mu \triangleright \lambda} \mathcal{B}_{r,n}^{\triangleright(f,\mu)}$, where in the sum $\mu \in \Lambda_r^+(n-2f)$.

By Definition 2.1, there is a natural homomorphism from $\mathcal{B}_{r,m}$ to $\mathcal{B}_{r,n}$ for positive integers $m \leq n$. Let $\mathcal{B}'_{r,m}$ be the image of $\mathcal{B}_{r,m}$ in $\mathcal{B}_{r,n}$. The following result, which plays the key role, has been proved by Yu without assuming that ω_0 is invertible [20].

Lemma 2.11. *N is a right $\mathcal{B}_{r,n}$ -module if N is the R -submodule generated by $\mathcal{B}'_{r,n-2f} E^{f,n} T_d X^{\kappa_d}$, for all $d \in \mathcal{D}_{f,n}$ and $\kappa_d \in \mathbb{N}_r^{f,n}$.*

Proposition 2.12. (cf. [19, 5.10]) *Suppose that $\mathbf{s} \in \mathcal{T}^{std}(\lambda)$. We define $\Delta_{\mathbf{s}}(f, \lambda)$ to be the R -submodule of $\mathcal{B}_{r,n}^{\triangleright(f,\lambda)} / \mathcal{B}_{r,n}^{\triangleright(f,\lambda)}$ spanned by*

$$\{ E^{f,n} M_{\mathbf{st}} T_d X^{\kappa_d} + \mathcal{B}_{r,n}^{\triangleright(f,\lambda)} \mid (\mathbf{t}, d, \kappa_d) \in \delta(f, \lambda) \},$$

where $\delta(f, \lambda) = \{ (\mathbf{t}, d, \kappa_d) \mid \mathbf{t} \in \mathcal{T}^{std}(\lambda), d \in \mathcal{D}_{f,n} \text{ and } \kappa_d \in \mathbb{N}_r^{f,n} \}$. Then $\Delta_{\mathbf{s}}(f, \lambda)$ is a right $\mathcal{B}_{r,n}$ -module.

Proof. By Lemma 2.11, $E^{f,n} M_{\mathbf{st}} T_d X^{\kappa_d} h + \mathcal{B}_{r,n}^{\triangleright(f,\lambda)}$ can be written as an R -linear combination of elements $M_{\mathbf{st}} \mathcal{B}'_{r,n-2f} E^{f,n} T_e X^{\kappa_e} + \mathcal{B}_{r,n}^{\triangleright(f,\lambda)}$ for $e \in \mathcal{D}_{f,n}$ and $\kappa_e \in \mathbb{N}_r^{f,n}$. By [19, 5.8d],

$$M_{\mathbf{st}} \mathcal{B}'_{r,n-2f} E^{f,n} \equiv E^{f,n} \varepsilon_{n-2f}(\mathbf{m}_{\mathbf{st}} \mathcal{H}_{r,n-2f}) \pmod{\mathcal{B}_{r,n}^{\triangleright(f,\lambda)}},$$

where $\mathbf{m}_{\mathbf{st}}$ is given in (2.8). Finally, using Dipper-James-Mathas's result on Murphy basis for Ariki-Koike algebras in [7] yields

$$M_{\mathbf{st}} \mathcal{B}'_{r,n-2f} E^{f,n} T_e X^{\kappa_e} + \mathcal{B}_{r,n}^{\triangleright(f,\lambda)} \in \Delta_{\mathbf{s}}(f, \lambda).$$

So, $\Delta_{\mathbf{s}}(f, \lambda)$ is a right $\mathcal{B}_{r,n}$ -module. \square

We recall the definition of cellular algebras in [11].

Definition 2.13. [11] Let R be a commutative ring and A an R -algebra. Fix a partially ordered set $\Lambda = (\Lambda, \triangleright)$ and for each $\lambda \in \Lambda$ let $T(\lambda)$ be a finite set. Finally, fix $C_{\mathbf{st}}^\lambda \in A$ for all $\lambda \in \Lambda$ and $\mathbf{s}, \mathbf{t} \in T(\lambda)$.

Then the triple (Λ, T, C) is a *cell datum* for A if:

- a) $\{ C_{\mathbf{st}}^\lambda \mid \lambda \in \Lambda \text{ and } \mathbf{s}, \mathbf{t} \in T(\lambda) \}$ is an R -basis for A ;
- b) the R -linear map $*$: $A \rightarrow A$ determined by $(C_{\mathbf{st}}^\lambda)^* = C_{\mathbf{ts}}^\lambda$, for all $\lambda \in \Lambda$ and all $\mathbf{s}, \mathbf{t} \in T(\lambda)$ is an anti-isomorphism of A ;
- c) for all $\lambda \in \Lambda$, $\mathbf{s} \in T(\lambda)$ and $a \in A$ there exist scalars $r_{\mathbf{tu}}(a) \in R$ such that

$$C_{\mathbf{st}}^\lambda a = \sum_{\mathbf{u} \in T(\lambda)} r_{\mathbf{tu}}(a) C_{\mathbf{su}}^\lambda \pmod{A^{\triangleright \lambda}},$$

where $A^{\triangleright \lambda} = R\text{-span} \{ C_{\mathbf{uv}}^\mu \mid \mu \triangleright \lambda \text{ and } \mathbf{u}, \mathbf{v} \in T(\mu) \}$.

Furthermore, each scalar $r_{\mathbf{tu}}(a)$ is independent of \mathbf{s} . An algebra A is a *cellular algebra* if it has a cell datum and in this case we call $\{ C_{\mathbf{st}}^\lambda \mid \mathbf{s}, \mathbf{t} \in T(\lambda), \lambda \in \Lambda \}$ a *cellular basis* of A .

Theorem 2.14. *Let $\mathcal{B}_{r,n}$ be the cyclotomic Birman-Wenzl algebras over R . Then*

$$\mathcal{C} = \bigcup_{(f,\lambda) \in \Lambda_{r,n}^+} \{ C_{(\mathbf{s},e,\kappa_e)(\mathbf{t},d,\kappa_d)}^{(f,\lambda)} \mid (\mathbf{s}, e, \kappa_e), (\mathbf{t}, d, \kappa_d) \in \delta(f, \lambda) \}$$

is a cellular basis of $\mathcal{B}_{r,n}$ where $C_{(\mathbf{s}, \mathbf{e}, \kappa_e)(\mathbf{t}, d, \kappa_d)}^{(f, \lambda)} = X^{\kappa_e} T_e^* E^{f, n} M_{\mathbf{st}} T_d X^{\kappa_d}$. The R -linear map $*$, which fixes $T_i, X_j, 1 \leq i \leq n-1$ and $1 \leq j \leq n$ is the required anti-involution. In particular, the rank of $\mathcal{B}_{r,n}$ is $r^n(2n-1)!!$.

Proof. This result can be proved by arguments in the proof of [19, 5.41]. We leave the details to the reader. The only difference is that we have to use Proposition 2.12 instead of [19, 5.10]. Finally, we remark that we use seminormal representations for $\mathcal{B}_{r,n}$ in the proof of [19, 5.41]. Such representations have been constructed in [19, 4.19] for all positive integers r . \square

Remark 2.15. Yu [20] has proved that $\mathcal{B}_{r,n}$ is cellular under the assumption that ω_0 is invertible. Finally, we remark that Theorem 2.14 for all odd positive integers r has been proved in [19, 5.41].

Let F be an arbitrary field, which contains the non-zero parameters q, u_1, \dots, u_r and $q - q^{-1}$. Assume that $\Omega \cup \{q\} \subset F$ is \mathbf{u} -admissible in the sense of the Assumption 2.2. We always keep this assumption when we consider $\mathcal{B}_{r,n}$ over F later on. Let $\mathcal{B}_{r,n,F}$ be the cyclotomic Birman–Wenzl algebra over F . By standard arguments, we have

$$\mathcal{B}_{r,n,F} \cong \mathcal{B}_{r,n} \otimes_R F.$$

In the remainder of this paper, we use $\mathcal{B}_{r,n}$ instead of $\mathcal{B}_{r,n,F}$ if there is no confusion.

By using Dipper–Mathas’s Morita equivalent theorem for Ariki–Koike algebras [8], we can assume $u_i = q^{k_i}, k_i \in \mathbb{Z}$ in the following theorem without loss of generality. See the remark in [4, p130].

Theorem 2.16. *Let $\mathcal{B}_{r,n}$ be the cyclotomic Birman–Wenzl algebra over F .*

- a) *If n is odd, then the non-isomorphic irreducible $\mathcal{B}_{r,n}$ -modules are indexed by (f, λ) where $0 \leq f \leq \lfloor \frac{n}{2} \rfloor$ and λ are \mathbf{u} -Kleshchev multipartitions of $n - 2f$ in the sense of [3].*
- b) *Suppose that n is an even number.*
 - (i) *If $\omega_i \neq 0$ for some non-negative integers $i \leq r - 1$, then the non-isomorphic irreducible $\mathcal{B}_{r,n}$ -modules are indexed by (f, λ) where $0 \leq f \leq \frac{n}{2}$ and λ are \mathbf{u} -Kleshchev multipartitions of $n - 2f$.*
 - (ii) *If $\omega_i = 0$ for all non-negative integers $i \leq r - 1$, then the set of all pair-wise non-isomorphic irreducible $\mathcal{B}_{r,n}$ -modules are indexed by (f, λ) where $0 \leq f < \frac{n}{2}$ and λ are \mathbf{u} -Kleshchev multipartitions of $n - 2f$.*

Proof. When r is odd, this is [19, 6.3]. In general, the result still follows from the arguments in [19, §6]. The reason why Rui and Xu had to assume that $2 \nmid r$ in [19, §6] is that they did not have Proposition 2.12 for $2 \mid r$ in [19]. We leave the details to the reader. \square

We close this section by giving a criterion on $\mathcal{B}_{r,n}$ being quasi-hereditary in the sense of [6].

Corollary 2.17. *Suppose that $\mathcal{B}_{r,n}$ is defined over the field F .*

- a) *Suppose that $\omega_i \neq 0$ for some $i, 0 \leq i \leq r - 1$. Then $\mathcal{B}_{r,n}$ is quasi-hereditary if and only if $o(q^2) > n$ and $|d| \geq n$ whenever $u_i u_j^{-1} - q^{2d} = 0$ and $d \in \mathbb{Z}$ with $1 \leq i \neq j \leq r$.*
- b) *Suppose that $\omega_i = 0$ for all $i, 0 \leq i \leq r - 1$. Then $\mathcal{B}_{r,n}$ is quasi-hereditary if and only if n is odd and $o(q^2) > n$ and $|d| \geq n$ whenever $u_i u_j^{-1} - q^{2d} = 0$ and $d \in \mathbb{Z}$ with $1 \leq i \neq j \leq r$.*

Proof. Note that $\mathcal{B}_{r,n}$ is cellular. By [11, 3.10], $\mathcal{B}_{r,n}$ is quasi-hereditary if and only if the non-isomorphic irreducible $\mathcal{B}_{r,n}$ -modules are indexed by $\Lambda_{r,n}^+$. So, the result follows from Theorem 2.16. In this case, the Ariki–Koike algebras $\mathcal{H}_{r,n-2f}$, $0 \leq f \leq \lfloor n/2 \rfloor$ are semisimple. \square

3. THE JM-BASIS OF $\Delta(f, \lambda)$

Throughout this section, we assume that $\mathcal{B}_{r,n}$ is defined over a commutative R . The main purpose of this section is to construct the JM-basis for $\mathcal{B}_{r,n}$.

Lemma 3.1. *Suppose that $n \geq 2$. We have $E_{n-1}\mathcal{B}_{r,n}E_{n-1} = E_{n-1}\mathcal{B}_{r,n-2}$.*

Proof. Since $\mathcal{B}_{r,n-2}E_{n-1} = E_{n-1}\mathcal{B}_{r,n-2}E_{n-2}E_{n-1} \subset E_{n-1}\mathcal{B}_{r,n}E_{n-1}$, we need only to show the inverse inclusion.

By Lemma 2.11 for $f = 1$, we need only prove that $E_{n-1}hE_{n-1} \in \mathcal{B}_{r,n-2}E_{n-1}$ for $h = T_d X^{\kappa_d}$ and $d \in \mathcal{D}_{1,n}$. By Definition 2.1(b)(c), we can assume $X^{\kappa_d} = X_{n-1}^k$ for some $k \in \mathbb{Z}$ without loss of generality.

Note that the Birman-Wenzl algebra $\mathcal{B}_{1,n}$ is a subalgebra of $\mathcal{B}_{r,n}$. The result for $k = 0$ follows from the corresponding result for $\mathcal{B}_{1,n}$ in [5]. Assume that $k \neq 0$. We have $i_1 = n - 1$ and $j_1 = n$ if $d = s_{n-1,i_1}s_{n,j_1}$. So, $d = 1$. By [19, 4.21], $E_{n-1}X_{n-1}^kE_{n-1} = \omega_{n-1}^{(k)}E_{n-1}$ for some $\omega_{n-1}^{(k)} \in \mathcal{B}_{r,n-2}$. So, $E_{n-1}\mathcal{B}_{r,n}E_{n-1} \subseteq E_{n-1}\mathcal{B}_{r,n-2}$. \square

Using Lemma 3.1 repeatedly yields the following result.

Corollary 3.2. $E^{f,n}\mathcal{B}_{r,n}E^{f,n} = \mathcal{B}_{r,n-2f}E^{f,n}$, for all positive integers $f \leq \lfloor \frac{n}{2} \rfloor$.

By Theorem 2.14, $\mathcal{B}_{r,n}$ is cellular over the poset $\Lambda_{r,n}^+$ in the sense of [11]. For each $(f, \lambda) \in \Lambda_{r,n}^+$, we have the cell module $\Delta(f, \lambda)$ with respect to the cellular basis of $\mathcal{B}_{r,n}$ given in Theorem 2.14. By definition, it is a right $\mathcal{B}_{r,n}$ -module which is isomorphic to $\Delta_{\mathbf{s}}(f, \lambda)$ defined in Proposition 2.12. Later on, we will identify $\Delta(f, \lambda)$ with $\Delta_{\mathbf{s}}(f, \lambda)$ for $\mathbf{s} = \mathbf{t}^\lambda$. We are going to construct a $\mathcal{B}_{r,n-1}$ -filtration of $\Delta(f, \lambda)$ by using arguments in [18].

Let $\sigma_f : \mathcal{H}_{r,n-2f} \longrightarrow \mathcal{B}_{r,n}^f / \mathcal{B}_{r,n}^{f+1}$ be the R -linear map defined by

$$(3.3) \quad \sigma_f(h) = E^{f,n}\varepsilon_{n-2f}(h) + \mathcal{B}_{r,n}^{f+1}$$

for all $h \in \mathcal{H}_{r,n-2f}$, $1 \leq f \leq \lfloor \frac{n}{2} \rfloor$. Here $\varepsilon_{n-2f} : \mathcal{H}_{r,n-2f} \rightarrow \mathcal{B}_{r,n-2f} / \mathcal{E}_{n-2f}$ is the algebraic isomorphism mentioned in section 2.

Given $\lambda \in \Lambda_r^+(n)$ and $\mu \in \Lambda_r(n)$. A λ -tableau \mathbf{S} is of type μ if it is obtained from $Y(\lambda)$ by inserting the entries (k, i) with $i \geq 1$ and $1 \leq k \leq r$ such that the number of the entries in \mathbf{S} which are equal to (k, i) is $\mu_i^{(k)}$.

For any $\mathbf{s} \in \mathcal{T}^{std}(\lambda)$, let $\mu(\mathbf{s})$ be obtained from \mathbf{s} by replacing each entry m in \mathbf{s} by (k, i) if m is in row i of the k -th component of \mathbf{t}^μ . Then $\mu(\mathbf{s})$ is a λ -tableau of type μ .

Given (k, i) and (ℓ, j) in $\{1, 2, \dots, r\} \times \mathbb{N}$, we say that $(k, i) < (\ell, j)$ if either $k < \ell$ or $k = \ell$ and $i < j$. In other words, $<$ is the lexicographic order on $\{1, 2, \dots, r\} \times \mathbb{N}$.

Following [7], we say that $\mathbf{S} = (\mathbf{S}^{(1)}, \mathbf{S}^{(2)}, \dots, \mathbf{S}^{(r)})$, a λ -tableau of type μ , is semi-standard if

- a) the entries in each row of each component $\mathbf{S}^{(k)}$ of \mathbf{S} increase weakly,
- b) the entries in each column of each component $\mathbf{S}^{(k)}$ of \mathbf{S} increase strictly,
- c) for each positive integer $k \leq r$ no entry in $\mathbf{S}^{(k)}$ is of form (ℓ, i) with $\ell < k$.

Let $\mathcal{T}^{ss}(\lambda, \mu)$ be the set of all semi-standard λ -tableaux of type μ . Given $\mathbf{S} \in \mathcal{T}^{ss}(\lambda, \mu)$ and $\mathbf{t} \in \mathcal{T}^{std}(\lambda)$. Motivated by [7], write

$$(3.4) \quad M_{\mathbf{S}\mathbf{t}} = \sum_{\substack{\mathbf{s} \in \mathcal{T}^{std}(\lambda) \\ \mu(\mathbf{s}) = \mathbf{S}}} M_{\mathbf{s}\mathbf{t}}.$$

Lemma 3.5. (cf. [18, 4.8, 4.11-4.13])

a) For any $h \in \mathcal{B}_{r,n}$, we have

$$E^{f,n}h \equiv \sum_{\substack{h_1 \in \mathcal{H}_{r,n-2f}^e \\ d \in \mathcal{D}_{f,n}^{f,n} \\ \kappa_d \in \mathbb{N}_{r,n}^{f,n}}} \sigma_f(h_1) T_d X^{\kappa_d} \pmod{\mathcal{B}_{r,n}^{f+1}}.$$

b) For each $\mu \in \Lambda_r^+(n-2f)$, let L^μ be the right $\mathcal{B}_{r,n}$ -submodule of $\mathcal{B}_{r,n}^f / \mathcal{B}_{r,n}^{f+1}$ generated by $E^{f,n}M_\mu \pmod{\mathcal{B}_{r,n}^{f+1}}$. Then L^μ is the free R -module generated by $\Upsilon = \{E^{f,n}M_{\text{St}T_d X^{\kappa_d}} \pmod{\mathcal{B}_{r,n}^{f+1}} \mid \mathbf{S} \in \mathcal{T}^{ss}(\lambda, \mu), (\mathbf{t}, d, \kappa_d) \in \delta(f, \lambda), \lambda \in \Lambda_r^+(n-2f)\}$.

c) Suppose that $(f, \lambda) \in \Lambda_{r,n}^+$ with $f > 0$. If $\mathbf{s} \in \mathcal{T}^{std}(\mu)$ such that $\mu \in \Lambda_r^+(n-2f+1)$ and $\tau = \mathbf{s}_{n-2f} \triangleright \lambda$, then $E^{f,n}T_{n-1,n-2f+1}T_{d(\mathbf{s})}^* M_\mu \in \mathcal{B}_{r,n}^{\triangleright(f,\lambda)}$.

d) Suppose that $(f, \lambda) \in \Lambda_{r,n}^+$ with $f > 0$ and $h \in E^{f-1,n-1}M_\lambda \mathcal{B}_{r,n-1} \cap \mathcal{B}_{r,n-1}^f$. Then

$$E_{n-1}T_{n-1,n-2f+1}h \equiv \sum_{\substack{h_1 \in \mathcal{H}_{r,n-2f}^e \\ e \in \mathcal{D}_{f,n-1}^{f,n-1} \\ \kappa_e \in \mathbb{N}_{r,n-1}^{f,n-1}}} E^{f,n}M_\lambda \varepsilon_{n-2f}(h_1) T_{n-2f,n} T_e X^{\kappa_e} \pmod{\mathcal{B}_{r,n}^{f+1}}.$$

Proof. One can use arguments in the proof of [18, 4.8] together with Corollary 3.2 to verify (a). (b)-(d) can be proved by arguments in the proof of [18, 4.11-4.13]. \square

Given two multipartitions λ and μ . We say that μ is obtained from λ by adding a box (or node) and write $\lambda \rightarrow \mu$ if there exists a pair (s, i) such that $\mu_i^{(s)} = \lambda_i^{(s)} + 1$ and $\mu_j^{(t)} = \lambda_j^{(t)}$ for $(t, j) \neq (s, i)$. In this case, we will also say that λ is obtained from μ by removing a box (or node).

Definition 3.6. Suppose $\lambda \in \Lambda_r^+(n-2f)$ with s removable nodes p_1, p_2, \dots, p_s and $m-s$ addable nodes $p_{s+1}, p_{s+2}, \dots, p_m$.

- Let $\mu_\lambda(i) \in \Lambda_r^+(n-2f-1)$ be obtained from λ by removing the box p_i for $1 \leq i \leq s$.
- Let $\mu_\lambda(j) \in \Lambda_r^+(n-2f+1)$ be obtained from λ by adding the box p_j for $s+1 \leq j \leq m$.

We identify $\mu_\lambda(i)$ with $(f, \mu_\lambda(i)) \in \Lambda_{r,n-1}^+$ (resp. $(f-1, \mu_\lambda(i)) \in \Lambda_{r,n-1}^+$) for $1 \leq i \leq s$ (resp. $s+1 \leq i \leq m$). So, $\mu_\lambda(i) \triangleright \mu_\lambda(j)$ for all i, j with $1 \leq i \leq s$ and $s+1 \leq j \leq m$. We arrange the nodes $p_i, 1 \leq i \leq m$ such that

$$(3.7) \quad \mu_\lambda(i) \triangleright \mu_\lambda(i+1) \quad \text{for all } i, 1 \leq i \leq m-1$$

with respect to the partial order \leq on $\Lambda_{r,n-1}^+$.

For each $\lambda = (\lambda^{(1)}, \lambda^{(2)}, \dots, \lambda^{(r)}) \in \Lambda_r^+(n)$, let $[\lambda] = [a_1, a_2, \dots, a_r]$ such that $a_i = \sum_{j=1}^i |\lambda^{(j)}|$, $1 \leq i \leq r$. Write $[\mu_\lambda(i)] = [b_1, b_2, \dots, b_r]$ for $s+1 \leq i \leq m$. In the later case, $\mu_\lambda(i)$ is obtained from λ by adding a box, say $p_i = (t, k, \lambda_k^{(t)} + 1)$. We remark that $(t, k, \ell) \in Y(\lambda)$ is in the k -th row, ℓ -th column of the t -th component of $Y(\lambda)$. When $1 \leq i \leq s$, $\mu_\lambda(i)$ is obtained from λ by removing the box, say $p_i = (t, k, \lambda_k^{(t)})$. We define

$$(3.8) \quad \begin{cases} a_{p_i} = a_{t-1} + \sum_{j=1}^k \lambda_j^{(t)}, & \text{if } 1 \leq i \leq s, \\ b_{p_i} = b_{t-1} + \sum_{j=1}^k \mu_\lambda(i)_j^{(t)}, & \text{if } s+1 \leq i \leq m, \end{cases}$$

and

$$(3.9) \quad y_{\mu_\lambda(i)}^\lambda = \begin{cases} E^{f,n}M_\lambda T_{a_{p_i}, n}, & \text{if } 1 \leq i \leq s, \\ E_{n-1}T_{n-1, b_{p_i}} E^{f-1, n-1}M_{\mu_\lambda(i)}, & \text{if } s+1 \leq i \leq m. \end{cases}$$

For each positive integer $i \leq m$, define

$$(3.10) \quad \delta(\lambda, i) = \{(\mathbf{s}, d, \kappa_d) \mid \mathbf{s} \in \mathcal{T}^{std}(\mu_\lambda(i)), d \in \mathcal{D}_{\ell, n-1}, \text{ and } \kappa_d \in \mathbb{N}_r^{\ell, n-1}\},$$

where $\ell = f$ (resp. $\ell = f - 1$) if $1 \leq i \leq s$ (resp. $s + 1 \leq i \leq m$).

In the remainder of this section, we will keep our previous notation $\mu_\lambda(i)$. In other words, $\mu_\lambda(i)$ is obtained from λ by removing (resp. adding) the node p_i for $1 \leq i \leq s$ (resp. $s + 1 \leq i \leq m$).

Theorem 3.11. *For any $(f, \lambda) \in \Lambda_{r,n}^+$ with $f \geq 0$, let $S^{\triangleright \mu_\lambda(i)}$ be the R -submodule of $\Delta(f, \lambda)$ generated by $\{y_{\mu_\lambda(j)}^\lambda T_{d(\mathbf{t})} T_d X^{\kappa_d} \pmod{\mathcal{B}_{r,n}^{\triangleright(f, \lambda)}} \mid (\mathbf{t}, d, \kappa_d) \in \delta(\lambda, j), 1 \leq j \leq i\}$. Then*

$$(0) \subseteq S^{\triangleright \mu_\lambda(1)} \subseteq \dots \subseteq S^{\triangleright \mu_\lambda(m)} = \Delta(f, \lambda)$$

is a $\mathcal{B}_{r,n-1}$ -filtration of $\Delta(f, \lambda)$. Further, we have the following $\mathcal{B}_{r,n-1}$ -isomorphism:

$$\Delta(\ell, \mu_\lambda(i)) \cong S^{\triangleright \mu_\lambda(i)} / S^{\triangleright \mu_\lambda(i-1)}, 1 \leq i \leq m.$$

Proof. When $f = 0$, each cell module $\Delta(0, \lambda)$ can be considered as a cell module for $\mathcal{H}_{r,n}$. The result for $f = 0$ has been given in [3]. In the remainder of the proof, we assume $f > 0$.

Using arguments in the proof of [18, 4.9, 4.14], we can prove that all $S^{\triangleright \mu_\lambda(i)}$, $1 \leq i \leq m$, are $\mathcal{B}_{r,n-1}$ -modules. Of course, we have to use Lemma 3.5 instead of [18, 4.8, 4.11-4.13]. So, $(0) \subseteq S^{\triangleright \mu_\lambda(1)} \subseteq \dots \subseteq S^{\triangleright \mu_\lambda(m)}$ is a filtration of $\mathcal{B}_{r,n-1}$ -modules.

Let $\phi_i : \Delta(\ell, \mu_\lambda(i)) \rightarrow S^{\triangleright \mu_\lambda(i)} / S^{\triangleright \mu_\lambda(i-1)}$ be the R -linear map sending $E^{\ell, n-1} M_{\mu_\lambda(i)} T_{d(\mathbf{t})} T_e X^{\kappa_e} \pmod{\mathcal{B}_{r,n-1}^{\triangleright(\ell, \mu_\lambda(i))}}$ to $y_{\mu_\lambda(i)}^\lambda T_{d(\mathbf{t})} T_e X^{\kappa_e} \pmod{S^{\triangleright \mu_\lambda(i-1)}}$ for all $(\mathbf{t}, e, \kappa_e) \in \delta(\lambda, i)$. ϕ_i is a $\mathcal{B}_{r,n-1}$ -homomorphism since multiplying an element on the left is a homomorphism of right modules.

We claim $\Delta(f, \lambda) = S^{\triangleright \mu_\lambda(m)}$. In fact, by Proposition 2.12 for $\mathbf{s} = \mathbf{t}^\lambda$ and Definition 2.1(h), $\Delta(f, \lambda) \subset E^{f,n} M_\lambda \mathcal{B}_{r,n-1} \pmod{\mathcal{B}_{r,n}^{\triangleright(f, \lambda)}}$. Note that $M_{\mu_\lambda(m)} = M_\lambda$. We have

$$\begin{aligned} E^{f,n} M_\lambda &= E_{n-1} T_{n-1, n-2f+1} E^{f-1, n-1} M_{\mu_\lambda(m)} T_{n-1, n-2f+1}^* \\ &= y_{\mu_\lambda(m)}^\lambda T_{n-1, n-2f+1}^* \in S^{\triangleright \mu_\lambda(m)}. \end{aligned}$$

Since $S^{\triangleright \mu_\lambda(m)}$ is a right $\mathcal{B}_{r,n-1}$ -module, $\Delta(f, \lambda) \subseteq S^{\triangleright \mu_\lambda(m)}$. The inverse inclusion is trivial. This proves our claim. Counting the rank of $\Delta(f, \lambda)$ forces each ϕ_i to be an R -linear isomorphism. \square

We are going to recall the notion of n -updown tableaux in [4] in order to construct the JM-basis of $\mathcal{B}_{r,n}$.

Fix $(f, \lambda) \in \Lambda_{r,n}^+$. An n -updown λ -tableau, or more simply an updown λ -tableau, is a sequence $\mathbf{t} = (\mathbf{t}_0, \mathbf{t}_1, \mathbf{t}_2, \dots, \mathbf{t}_n)$ of multipartitions such that $\mathbf{t}_0 = \emptyset$, $\mathbf{t}_n = \lambda$ and \mathbf{t}_i is obtained from \mathbf{t}_{i-1} by either adding or removing a box, for $i = 1, \dots, n$. Let $\mathcal{T}_n^{ud}(\lambda)$ be the set of all n -updown λ -tableaux.

Given $\mathbf{t} \in \mathcal{T}_n^{ud}(\lambda)$ with $(f, \lambda) \in \Lambda_{r,n}^+$, define $f_j \in \mathbb{N}$ by declaring that $\mathbf{t}_j \in \Lambda_r^+(j - 2f_j)$. So, $0 \leq f_j \leq \lfloor \frac{j}{2} \rfloor$.

Motivated by [18], we define $\mathbf{m}_{\mathbf{t}} = \mathbf{m}_{\mathbf{t}_n} \in \mathcal{B}_{r,n}$ inductively by declaring that $\mathbf{m}_{\mathbf{t}_0} = 1$ and

- a) $\mathbf{m}_{\mathbf{t}_i} = \sum_{j=a_{s,k}+1}^{a_{s,k}} q^{a_{s,k}-j} T_{j, a_s} \prod_{j=s}^{r-1} (X_{a_j} - u_{j+1}) T_{a_j, a_{j+1}} T_{a_r, i} \mathbf{m}_{\mathbf{t}_{i-1}}$ if $\mathbf{t}_i = \mathbf{t}_{i-1} \cup p$ with $p = (s, k, \mu_k^{(s)})$ and $a_{s,k} = a_{s-1} + \sum_{j=1}^k \mu_j^{(s)}$.
- b) $\mathbf{m}_{\mathbf{t}_i} = E_{i-1} T_{i-1, b_{s,k}} \mathbf{m}_{\mathbf{t}_{i-1}}$ if $\mathbf{t}_{i-1} = \mathbf{t}_i \cup p$ with $p = (s, k, \nu_k^{(s)})$ and $b_{s,k} = b_{s-1} + \sum_{j=1}^k \nu_j^{(s)}$.

where $\mu = \mathbf{t}_i$ and $\nu = \mathbf{t}_{i-1}$ with $[\mu] = [a_1, a_2, \dots, a_r]$ and $[\nu] = [b_1, b_2, \dots, b_r]$.
Now, we define $b_{\mathbf{t}_i}$ inductively such that

$$\mathbf{m}_{\mathbf{t}} \equiv E^{f,n} M_{\lambda} b_{\mathbf{t}_n} \pmod{\mathcal{B}_{r,n}^{\triangleright(f,\lambda)}}.$$

We write $\mathbf{m}_{\lambda} = E^{f,n} M_{\lambda}$. Suppose that $\mathbf{t}_{n-1} = \mu$, $[\lambda] = [a_1, a_2, \dots, a_r]$ and $[\mu] = [b_1, b_2, \dots, b_r]$. We have $b_{\mathbf{t}_0} = 1$ and

$$(3.12) \quad b_{\mathbf{t}_n} = \begin{cases} T_{a_{\ell,k},n} b_{\mathbf{t}_{n-1}}, & \text{if } \mathbf{t}_n = \mathbf{t}_{n-1} \cup \{(\ell, k, \lambda_k^{(\ell)})\}, \\ T_{n-1, b_{r-1}} h b_{\mathbf{t}_{n-1}}, & \text{if } \mathbf{t}_{n-1} = \mathbf{t}_n \cup \{(s, k, \mu_k^{(s)})\}, \\ T_{n-1, b_{r,k}} \sum_{j=b_{r,k-1}+1}^{b_{r,k}} q^{b_{r,k}-j} T_{b_{r,k},j} b_{\mathbf{t}_{n-1}}, & \text{if } \mathbf{t}_{n-1} = \mathbf{t}_n \cup \{(r, k, \mu_k^{(r)})\}, \end{cases}$$

where $s \neq r$ and

$$h = \prod_{j=r}^{s+2} \{(X_{b_{j-1}} - u_j) T_{b_{j-1}, b_{j-2}}\} \times (X_{b_s} - u_{s+1}) T_{b_s, b_{s,k}} \sum_{j=b_{s,k-1}+1}^{b_{s,k}} q^{b_{s,k}-j} T_{b_{s,k},j}.$$

We also use $b_{\mathbf{t}}$ instead of $b_{\mathbf{t}_n}$.

For any $\mathbf{s}, \mathbf{t} \in \mathcal{T}_n^{ud}(\lambda)$, we identify \mathbf{s}_i (resp. \mathbf{t}_i) with (f_i, \mathbf{s}_i) (resp. (g_i, \mathbf{t}_i)) where $(f_i, \mathbf{s}_i), (g_i, \mathbf{t}_i) \in \Lambda_{r,i}^+$. We write $\mathbf{s} \succ^k \mathbf{t}$ if $\mathbf{s}_j \triangleright \mathbf{t}_j$ and $\mathbf{s}_l = \mathbf{t}_l$ for $j+1 \leq l \leq n$ and $j \geq k$. We write $\mathbf{s} \succ \mathbf{t}$ if there is a positive integer $k \leq n-1$ such that $\mathbf{s} \succ^k \mathbf{t}$. In [18], we have verified that $\mathbf{s} \succ \mathbf{v}$ if $\mathbf{s} \succ \mathbf{t}$ and $\mathbf{t} \succ \mathbf{v}$. So, \succ can be refined to be a linear order on $\mathcal{T}_n^{ud}(\lambda)$.

There is a partial order \triangleright on $\mathcal{T}_n^{ud}(\lambda)$. More explicitly, we have $\mathbf{s} \triangleright \mathbf{t}$ if $\mathbf{s}_i \triangleright \mathbf{t}_i$, $1 \leq i \leq n$. We write $\mathbf{s} \triangleright \mathbf{t}$ if $\mathbf{s} \triangleright \mathbf{t}$ and $\mathbf{s} \neq \mathbf{t}$.

There is a unique element, say $\mathbf{t}^{\lambda} \in \mathcal{T}_n^{ud}(\lambda)$, which is maximal with respect to \succ . More explicitly, we have $\mathbf{t}_{2i}^{\lambda} = \emptyset$ and $\mathbf{t}_{2i-1}^{\lambda} = ((1), \emptyset, \dots, \emptyset)$ for $1 \leq i \leq f$ and $\mathbf{t}_j^{\lambda} = \mathbf{t}_{j-2f}^{\lambda}$ for $2f+1 \leq j \leq n$.

Let $E_{i,j} = E_i E_{i+1} \cdots E_{j-1}$ for $i < j$. If $i = j$, we set $E_{ij} = 1$. When $i > j$, we define $E_{i,j} = E_{i-1} E_{i-2} \cdots E_j$. So,

$$(3.13) \quad \mathbf{m}_{\mathbf{t}^{\lambda}} = E^{f,n} M_{\lambda} \prod_{i=1}^f E_{n-2(f-i)-1, 2i-1} \prod_{j=2}^r \prod_{k=1}^f (X_{2k-1} - u_j).$$

Suppose $\mathbf{t} \in \mathcal{T}_n^{ud}(\lambda)$ with $(f, \lambda) \in \Lambda_{r,n}^+$. Let

$$(3.14) \quad c_{\mathbf{t}}(k) = \begin{cases} u_s q^{2(j-i)}, & \text{if } \mathbf{t}_k = \mathbf{t}_{k-1} \cup (s, i, j), \\ u_s^{-1} q^{-2(j-i)}, & \text{if } \mathbf{t}_{k-1} = \mathbf{t}_k \cup (s, i, j), \end{cases}$$

and

$$(3.15) \quad c_{\lambda}(p) = \begin{cases} u_s q^{2(j-i)}, & \text{if } p = (s, i, j) \text{ is an addable node of } \lambda, \\ u_s^{-1} q^{-2(j-i)}, & \text{if } p = (s, i, j) \text{ is a removable node of } \lambda. \end{cases}$$

In the remainder of this paper, unless otherwise stated, we always use $\mathbf{m}_{\mathbf{t}}$ instead of $\mathbf{m}_{\mathbf{t}} + \mathcal{B}_{r,n}^{\triangleright(f,\lambda)} \in \Delta(f, \lambda)$.

Proposition 3.16. *a) $\{\mathbf{m}_{\mathbf{t}} \mid \mathbf{t} \in \mathcal{T}_n^{ud}(\lambda)\}$ is an R -basis of $\Delta(f, \lambda)$ for any $(f, \lambda) \in \Lambda_{r,n}^+$.*

b) $\mathbf{m}_{\mathbf{t}} (\prod_{i=1}^n X_i) = \prod_{k=1}^n c_{\mathbf{t}^{\lambda}}(k) \mathbf{m}_{\mathbf{t}}, \forall \mathbf{t} \in \mathcal{T}_n^{ud}(\lambda)$.

Proof. (a) follows immediately from Theorem 3.11. In order to prove (b), we consider $\mathcal{B}_{r,n}$ over the field of fraction of R_0 where $R_0 = \mathbb{Z}[u_1^{\pm}, u_2^{\pm}, \dots, u_r^{\pm}, q^{\pm}, (q - q^{-1})^{-1}]$. Note that we are assuming that u_1, u_2, \dots, u_r, q are indeterminates. By the counterpart of [4, 5.3] for $\mathcal{B}_{r,n}$, we have that $\mathcal{B}_{r,n}$ is split semisimple. Therefore, each cell module of $\mathcal{B}_{r,n}$ is irreducible. In particular, $\Delta(f, \lambda)$ is irreducible. By Definition 2.1, we have that $\prod_{i=1}^n X_i$ is central in $\mathcal{B}_{r,n}$. By Schur's Lemma,

$\prod_{i=1}^n X_i$ acts on $\Delta(f, \lambda)$ as a scalar. This enables us to consider the special case $\mathbf{t} = \mathbf{t}^\lambda$ without loss of generality. By direct computation,

$$\mathbf{m}_{\mathbf{t}^\lambda} X_i = \begin{cases} u_1^{(-1)^{i-1}} \mathbf{m}_{\mathbf{t}^\lambda}, & \text{if } 1 \leq i \leq 2f, \\ c_{\mathbf{t}^\lambda}(i) \mathbf{m}_{\mathbf{t}^\lambda}, & \text{if } 2f+1 \leq i \leq n. \end{cases}$$

So, $\mathbf{m}_{\mathbf{t}}(\prod_{i=1}^n X_i) = \prod_{k=1}^n c_{\mathbf{t}^\lambda}(k) \mathbf{m}_{\mathbf{t}}$. By (a), $\mathbf{m}_{\mathbf{t}}$ is an R_0 -basis. So (b) holds over R_0 . Finally, we use standard arguments on base change to get (b) over a commutative ring R . \square

Theorem 3.17. (cf. [18, 5.12]) Let $\mathbf{t} \in \mathcal{T}_n^{ud}(\lambda)$ with $(f, \lambda) \in \Lambda_{r,n}^+$. For any k , $1 \leq k \leq n$, there are some $\mathbf{u} \in \mathcal{T}_n^{ud}(\lambda)$ and $a_{\mathbf{u}} \in R$ such that

$$\mathbf{m}_{\mathbf{t}} X_k = c_{\mathbf{t}}(k) \mathbf{m}_{\mathbf{t}} + \sum_{\substack{\mathbf{u} \succ^{k-1} \mathbf{t}}} a_{\mathbf{u}} \mathbf{m}_{\mathbf{u}}.$$

Proof. Note that $\prod_{k=1}^n c_{\mathbf{t}}(k) = \prod_{k=1}^n c_{\mathbf{t}^\lambda}(k)$ for any $\mathbf{t} \in \mathcal{T}_n^{ud}(\lambda)$. By Lemma 3.16(b), $\mathbf{m}_{\mathbf{t}} \prod_{k=1}^n X_k = \prod_{k=1}^n c_{\mathbf{t}}(k) \mathbf{m}_{\mathbf{t}}$. We consider the action of $\prod_{i=1}^{n-1} X_i$ on $\mathbf{m}_{\mathbf{t}}$. We use the $\mathcal{B}_{r,n-1}$ -filtration of $\Delta(f, \lambda)$ in Theorem 3.11. By Lemma 3.16(b),

$$\mathbf{m}_{\mathbf{t}} \prod_{j=1}^{n-1} X_j - \prod_{j=1}^{n-1} c_{\mathbf{t}}(j) \mathbf{m}_{\mathbf{t}} \in S^{\geq \mu_\lambda(i-1)}$$

where $\mu_\lambda(j)$, $1 \leq j \leq m$ are defined in Theorem 3.11 with $\mu_\lambda(i) = \mathbf{t}_{n-1}$. Since $S^{\geq \mu_\lambda(i-1)}$ is a right $\mathcal{B}_{r,n-1}$ -module,

$$(3.18) \quad \mathbf{m}_{\mathbf{t}} X_n c_{\mathbf{t}}(n)^{-1} - \mathbf{m}_{\mathbf{t}} = \mathbf{m}_{\mathbf{t}} \prod_{j=1}^{n-1} c_{\mathbf{t}}(j) \prod_{j=1}^{n-1} X_j^{-1} - \mathbf{m}_{\mathbf{t}} \in S^{\geq \mu_\lambda(i-1)}.$$

So, Theorem 3.17 holds for $k = n$. When we deal with the case $k = n-1$, we consider the filtration of $\mathcal{B}_{r,n-2}$ -submodules of $S^{\geq \mu_\lambda(i)}/S^{\geq \mu_\lambda(i-1)}$. Note that $S^{\geq \mu_\lambda(i)}/S^{\geq \mu_\lambda(i-1)} \cong \Delta(\ell, \mu_\lambda(i))$ where $\Delta(\ell, \mu_\lambda(i))$ is the cell module for $\mathcal{B}_{r,n-1}$ with respect to $(\ell, \mu_\lambda(i)) \in \Lambda_{r,(n-1)}^+$. By similar arguments as above we can verify the result for $k = n-2$. Using these arguments repeatedly yields the required formula for general k . \square

Standard arguments prove the following result (cf. [16, 2.7]).

Theorem 3.19. For each $\mathbf{t}, \mathbf{s} \in \mathcal{T}_n^{ud}(\lambda)$ with $(f, \lambda) \in \Lambda_{r,n}^+$, let $\mathbf{m}_{\mathbf{s}\mathbf{t}} = b_{\mathbf{s}}^* \mathbf{m}_\lambda b_{\mathbf{t}}$, where $*$: $\mathcal{B}_{r,n} \rightarrow \mathcal{B}_{r,n}$ is the R -linear anti-involution which fixes the generators T_i, X_j for $1 \leq i \leq n-1$ and $1 \leq j \leq n$.

- a) $\mathcal{M} = \{\mathbf{m}_{\mathbf{s}\mathbf{t}} \mid \mathbf{s}, \mathbf{t} \in \mathcal{T}_n^{ud}(\lambda), (f, \lambda) \in \Lambda_{r,n}^+\}$ is a cellular basis of $\mathcal{B}_{r,n}$ over R .
- b) $\mathbf{m}_{\mathbf{s}\mathbf{t}} X_k \equiv c_{\mathbf{t}}(k) \mathbf{m}_{\mathbf{s}\mathbf{t}} + \sum_{\substack{\mathbf{u} \succ^{k-1} \mathbf{t}}} a_{\mathbf{u}} \mathbf{m}_{\mathbf{s}\mathbf{u}} \pmod{\mathcal{B}_{r,n}^{\triangleright(f,\lambda)}}.$

Remark 3.20. Note that \succ is a linear order on $\mathcal{T}_n^{ud}(\lambda)$. So, \mathcal{M} is a JM-basis and $\{X_1, \dots, X_n\}$ is a family of JM-element in the sense of [15, 2.4].

Given two partitions λ, μ , write $\lambda \ominus \mu$ if either $\lambda \subset \mu$ and $\mu \setminus \lambda = p$ for some removable node p of μ or $\lambda \supset \mu$ and $\lambda \setminus \mu = p$ for some removable node p of λ .

Given an $\mathbf{s} \in \mathcal{T}_n^{ud}(\lambda)$ and a positive integer $k < n$. If $\mathbf{s}_k \ominus \mathbf{s}_{k-1}$ and $\mathbf{s}_{k+1} \ominus \mathbf{s}_k$ are in different rows and in different columns then we define, following [4], $\mathbf{s}s_k$ to be the updown λ -tableau

$$\mathbf{s}s_k = (\mathbf{s}_1, \dots, \mathbf{s}_{k-1}, \mathbf{t}_k, \mathbf{s}_{k+1}, \dots, \mathbf{s}_n)$$

where \mathbf{t}_k is the multipartition which is uniquely determined by the conditions $\mathbf{t}_k \ominus \mathbf{s}_{k+1} = \mathbf{s}_{k-1} \ominus \mathbf{s}_k$ and $\mathbf{s}_{k-1} \ominus \mathbf{t}_k = \mathbf{s}_k \ominus \mathbf{s}_{k+1}$. If the nodes $\mathbf{s}_k \ominus \mathbf{s}_{k-1}$ and $\mathbf{s}_{k+1} \ominus \mathbf{s}_k$ are both in the same row, or both in the same column, then $\mathbf{s}s_k$ is not defined.

Lemma 3.21. (cf. [18, 5.13]) Suppose that $\mathfrak{t} \in \mathcal{T}_n^{ud}(\lambda)$ with $\mathfrak{t}_{i-2} \neq \mathfrak{t}_i$ and $\mathfrak{t}s_{i-1} \triangleleft \mathfrak{t}$.

- a) If $\mathfrak{t}_{i-2} \subset \mathfrak{t}_{i-1} \subset \mathfrak{t}_i$, then $\mathfrak{m}_{\mathfrak{t}}T_{i-1} = \mathfrak{m}_{\mathfrak{t}s_{i-1}} + \sum_{\mathfrak{u} \succ_{\mathfrak{t}s_{i-1}}^{i-1}} a_{\mathfrak{u}}\mathfrak{m}_{\mathfrak{u}}$ for some scalars $a_{\mathfrak{u}} \in R$.
- b) If $\mathfrak{t}_{i-2} \supset \mathfrak{t}_{i-1} \subset \mathfrak{t}_i$ such that $(\tilde{p}, \ell) > (p, k)$ where $\mathfrak{t}_{i-2} \setminus \mathfrak{t}_{i-1} = (p, k, \nu_k^{(p)})$, $\mathfrak{t}_i \setminus \mathfrak{t}_{i-1} = (\tilde{p}, \ell, \mu_{\ell}^{(\tilde{p})})$, $\mathfrak{t}_{i-2} = \nu$ and $\mathfrak{t}_i = \mu$, then $\mathfrak{m}_{\mathfrak{t}}T_{i-1}^{-1} = \mathfrak{m}_{\mathfrak{t}s_{i-1}} + \sum_{\mathfrak{u} \succ_{\mathfrak{t}s_{i-1}}^{i-1}} a_{\mathfrak{u}}\mathfrak{m}_{\mathfrak{u}}$ for some scalars $a_{\mathfrak{u}} \in R$.

Proof. First, we assume $i = n$. One can prove (a) by verifying $\mathfrak{m}_{\mathfrak{t}}T_{n-1} = \mathfrak{m}_{\mathfrak{t}s_{n-1}}$ via (3.12). We leave the details to the reader.

In order to prove (b), write $\mathfrak{t}_{n-2} \setminus \mathfrak{t}_{n-1} = (p, k, \nu_k^{(p)})$ and $\mathfrak{t}_n \setminus \mathfrak{t}_{n-1} = (\tilde{p}, \ell, \mu_{\ell}^{(\tilde{p})})$. Let $a = a_{\tilde{p}-1} + \sum_{i=1}^{\ell} \lambda_i^{(\tilde{p})}$, $c = c_{p-1} + \sum_{i=1}^k \nu_i^{(p)}$. Since either $\tilde{p} > p$ or $\tilde{p} = p$ and $\ell > k$, we have $a \geq c$.

First, we assume $p < r$, then

$$\mathfrak{m}_{\mathfrak{t}} = E^{f,n} M_{\lambda} T_{a,n} T_{n-2,c_{r-1}} Ab_{\mathfrak{t}_{n-2}} + \mathcal{B}_{r,n}^{\triangleright(f,\lambda)}$$

where

$$(3.22) \quad A = \prod_{j=r}^{p+2} (X_{c_{j-1}} - u_j) T_{c_{j-1},c_{j-2}} \times (X_{c_p} - u_{p+1}) T_{c_p,c} \sum_{j=c_{p,k-1}+1}^c q^{c-j} T_{c,j}.$$

We prove (b) by induction on \tilde{p} .

If $\tilde{p} = r$, then $a \geq c_{r-1}$. It is routine to verify $\mathfrak{m}_{\mathfrak{t}}T_{n-1}^{-1} = \mathfrak{m}_{\mathfrak{t}s_{n-1}}$.

If $\tilde{p} = r-1$, then $c_{r-2} \leq a \leq c_{r-1}$. We have

$$(3.23) \quad \begin{aligned} \mathfrak{m}_{\mathfrak{t}}T_{n-1}^{-1} &\equiv E^{f,n} M_{\lambda} T_{n-1,c_{r-1}+1} T_{a,c_{r-1}} \{ (X_{c_{r-1}+1} - u_r) T_{c_{r-1}+1,c_{r-2}} \\ &\quad + \delta X_{c_{r-1}+1} E_{c_{r-1}} T_{c_{r-1},c_{r-2}} - \delta X_{c_{r-1}+1} T_{c_{r-1},c_{r-2}} \} A \\ &\quad \times T_{c_{r-1}+1,n-1} b_{\mathfrak{t}_{n-2}} \pmod{\mathcal{B}_{r,n}^{\triangleright(f,\lambda)}} \end{aligned}$$

Since $T_{n-1,c_{r-1}+1} X_{c_{r-1}+1} T_{c_{r-1}+1,n-1} = X_{n-1}$, the third term on the right hand of (3.23) is equal to

$$h := \delta \sum_{j=a_{\tilde{p},\ell-1}+1}^a q^{a-j} T_{j,a} T_{a,c} E^{f,n} M_{\nu} b_{\mathfrak{t}_{n-2}} X_{n-1}$$

with $\nu = \mathfrak{t}_{n-2}$. Since we are assuming that $\nu \triangleright \lambda$, $h \in \mathcal{B}_{r,n}^{\triangleright(f,\lambda)}$.

The first term on the right hand side of the above equality is equal to $\mathfrak{m}_{\mathfrak{t}s_{n-1}}$. One can verify it by arguments in the proof of [18, 5.13]. We leave the details to the reader.

Finally we consider the second term h_1 on the right hand side of (3.23). Since $T_{a,c_{r-1}} X_{c_{r-1}}^{-1} = X_a^{-1} T_{c_{r-1},a}^{-1}$ and $E^{f,n} T_{n-1,c_{r-1}+1} E_{c_{r-1}} T_{c_{r-1}+1,n-1} = E^{f,n} T_{c_{r-1},n} T_{n-2,c_{r-1}}$, $\delta^{-1} h_1$ is equal to

$$\begin{aligned} &E^{f,n} M_{\lambda} X_a^{-1} T_{c_{r-1},a}^{-1} T_{c_{r-1},n} T_{n-2,c_{r-2}} Ab_{\mathfrak{t}_{n-2}} + \mathcal{B}_{r,n}^{\triangleright(f,\lambda)} \\ &= c_{\mathfrak{t}\lambda}(a)^{-1} E^{f,n} M_{\lambda} \prod_{j=a}^{c_{r-1}-1} (T_j - \delta) T_{c_{r-1},n} T_{n-2,c_{r-1}} T_{c_{r-1},c_{r-2}} \\ &\quad \times Ab_{\mathfrak{t}_{n-2}} + \mathcal{B}_{r,n}^{\triangleright(f,\lambda)}. \end{aligned}$$

Note that $\prod_{j=a}^{c_{r-1}-1} (T_j - \delta) \times T_{c_{r-1},n}$ can be written as an R -linear combination of $T_{\ell,n} h$, with $a \leq \ell \leq c_{r-1}$ and $h \in \mathcal{B}_{r,\ell-1}$. So $\delta^{-1} c_{\mathfrak{t}\lambda}(a) h_1$ can be written as an R -linear combination of the following elements

$$E^{f,n} M_{\lambda} T_{\ell,n} T_{n-2,c_{r-1}} T_{c_{r-1},c_{r-2}} Ab_{\mathfrak{t}_{n-2}} h + \mathcal{B}_{r,n}^{\triangleright(f,\lambda)}.$$

Note that $M_\lambda T_w \equiv q^{l(w)} M_\lambda \pmod{\langle E_1 \rangle}$ if $w \in \mathfrak{S}_\lambda$. So, $M_\lambda T_{\ell, n-2f} \equiv q^k M_\lambda T_{b, n-2f} \pmod{\langle E_1 \rangle}$ for some integers k, b such that $\mathbf{v} = \mathbf{t}^\lambda s_{b, n-2f}$ is a row standard tableau. Furthermore, since $b \geq \ell \geq a$, $\mathbf{v}_{n-2f-1} \supseteq \mathbf{t}_{n-1}$. If \mathbf{v} is not standard, we use [14, 3.15] and [19, 5.8] to get

$$E^{f,n} M_\lambda T_{\ell, n-2f} \equiv \sum_{\mathbf{s} \in \mathcal{T}^{std}(\lambda), \mathbf{s} \supseteq \mathbf{v}} a_{\mathbf{s}} E^{f,n} M_\lambda T_{d(\mathbf{s})} \pmod{\mathcal{B}_{r,n}^{\triangleright(f,\lambda)}}$$

for some scalars $a_{\mathbf{s}} \in R$. We write $d(\mathbf{s}) = s_{\ell', n-2f} d(\mathbf{s}')$ where \mathbf{s}' is obtained from \mathbf{s} by removing the entry $n-2f$. Since $\mathbf{s} \supseteq \mathbf{v}$, $\mathbf{s}' \in \mathcal{T}^{std}(\alpha)$ for $\alpha \in \Lambda_r^+(n-2f-1)$ with $\alpha \supseteq \mathbf{v}_{n-2f-1} \supseteq \mathbf{t}_{n-1} \supseteq (\mathbf{t}_{s_{n-1}})_{n-1}$. Therefore, h_1 can be written as an R -linear combination of the elements

$$E^{f,n} M_\lambda T_{\ell', n} T_{d(\mathbf{s}')} T_{n-2, c_{r-1}} T_{c_{r-1}, c_{r-2}} A b_{\mathbf{t}_{n-2}} h \pmod{\mathcal{B}_{r,n}^{\triangleright(f,\lambda)}}$$

Note that $E^{f,n} M_\lambda T_{\ell', n} = y_\alpha^\lambda$, and $\alpha = \mu_\lambda(i)$ for some $i, 1 \leq i \leq s$. So, the above element can be written as an R -linear combination of the elements in $\{\mathbf{m}_{\mathbf{s}} | \mathbf{s} \in \mathcal{T}_n^{ud}(\lambda), \mathbf{s}_{n-1} \supseteq \mathbf{t}_{n-1} \supseteq (\mathbf{t}_{s_{n-1}})_{n-1}\}$. In this case, $\mathbf{s} \succ^{n-1} \mathbf{t}_{s_{n-1}}$.

However, when $\tilde{p} < r-1$, the first term is not equal to $\mathbf{m}_{\mathbf{t}_{s_{n-1}}}$. We will use it instead of $\mathbf{m}_{\mathbf{t}_{n-1}} T_{n-1}^{-1}$ to get a similar equality for $i = c_{r-2}$. This will enable us to get three terms. If $\tilde{p} = r-2$, we will be done since the first term must be $\mathbf{m}_{\mathbf{t}_{s_{n-1}}}$. The second and the third term can be written as an R -linear combination of $\mathbf{m}_{\mathbf{u}}$ with $\mathbf{u} \succ^{n-1} \mathbf{t}_{s_{n-1}}$. In general, we have to repeat the above procedure to get the required formula. This completes the proof of our result under the assumption $p < r$.

Let $p = r$. Note that $a \geq c$. It is routine to check that

$$\mathbf{m}_{\mathbf{t}} T_{n-1}^{-1} \equiv \mathbf{m}_{\mathbf{t}_{s_{n-1}}} \pmod{\mathcal{B}_{r,n}^{\triangleright(f,\lambda)}}.$$

This completes the proof of the result for $i = n$. In general, we use Theorem 3.11 and the definition of \succ to reduce the result to the case for $i = n$. \square

4. RECURSIVE FORMULAE FOR GRAM DETERMINANTS

In this section, we assume that $\mathcal{B}_{r,n}$ is defined over a field F such that the following assumptions hold.

Assumption 4.1. Assume that $\mathbf{u} = (u_1, u_2, \dots, u_r) \in F^r$ is generic in the sense that $|d| \geq 2n$ whenever there exists $d \in \mathbb{Z}$ such that either $u_i u_j^{\pm 1} = q^{2d} 1_F$ and $i \neq j$, or $u_i = \pm q^d \cdot 1_F$. We will also assume $o(q^2) > n$.

Suppose that $\mathbf{s}, \mathbf{t} \in \mathcal{T}_n^{ud}(\lambda)$. Under the Assumption 4.1, Rui and Xu have proved that $\mathbf{s} = \mathbf{t}$ if and only if $c_{\mathbf{s}}(k) = c_{\mathbf{t}}(k)$, $1 \leq k \leq n$ [19, 4.5]. So, Assumption 4.1 is the **separate condition** in the sense of [15, 2.8]. This enables us to use standard arguments in [15] to construct an orthogonal basis for $\Delta(f, \lambda)$ as follows.

For each positive integer $k \leq n$, let

$$R(k) = \{c_{\mathbf{t}}(k) \mid \mathbf{t} \in \mathcal{T}_n^{ud}(\lambda)\}.$$

For $\mathbf{s}, \mathbf{t} \in \mathcal{T}_n^{ud}(\lambda)$, let

- a) $F_{\mathbf{t}} = \prod_{k=1}^n F_{\mathbf{t}, k}$,
- b) $f_{\mathbf{s}\mathbf{t}} = F_{\mathbf{s}} \mathbf{m}_{\mathbf{s}\mathbf{t}} F_{\mathbf{t}}$,
- c) $f_{\mathbf{s}} = \mathbf{m}_{\mathbf{s}} F_{\mathbf{s}} \pmod{\mathcal{B}_{r,n}^{\triangleright(f,\lambda)}}$,

where

$$(4.2) \quad F_{\mathbf{t}, k} = \prod_{\substack{r \in \mathcal{R}(k) \\ c_{\mathbf{t}}(k) \neq r}} \frac{X_k - r}{c_{\mathbf{t}}(k) - r}.$$

The following results hold for a general class of cellular algebras which have JM-bases such that the separate condition holds [15, §3].

Lemma 4.3. Suppose that $\mathbf{t} \in \mathcal{T}_n^{ud}(\lambda)$ with $(f, \lambda) \in \Lambda_{r,n}^+$.

- a) $f_{\mathbf{t}} = \mathbf{m}_{\mathbf{t}} + \sum_{\mathbf{s} \in \mathcal{T}_n^{ud}(\lambda)} a_{\mathbf{s}} \mathbf{m}_{\mathbf{s}}$, and $\mathbf{s} \succ \mathbf{t}$ if $a_{\mathbf{s}} \neq 0$.
- b) $\mathbf{m}_{\mathbf{t}} = f_{\mathbf{t}} + \sum_{\mathbf{s} \in \mathcal{T}_n^{ud}(\lambda)} b_{\mathbf{s}} f_{\mathbf{s}}$, and $\mathbf{s} \succ \mathbf{t}$ if $b_{\mathbf{s}} \neq 0$.
- c) $f_{\mathbf{t}} X_k = c_{\mathbf{t}}(k) f_{\mathbf{t}}$, for any integer k , $1 \leq k \leq n$.
- d) $f_{\mathbf{t}} F_{\mathbf{s}} = \delta_{\mathbf{s}\mathbf{t}} f_{\mathbf{t}}$ for all $\mathbf{s} \in \mathcal{T}_n^{ud}(\mu)$ with $(\frac{n-|\mu|}{2}, \mu) \in \Lambda_{r,n}^+$.
- e) $\{f_{\mathbf{t}} \mid \mathbf{t} \in \mathcal{T}_n^{ud}(\lambda)\}$ is a basis of $\Delta(f, \lambda)$.
- f) The Gram determinants associated to $\Delta(f, \lambda)$ defined by $\{f_{\mathbf{t}} \mid \mathbf{t} \in \mathcal{T}_n^{ud}(\lambda)\}$ and the JM-basis in Proposition 3.16 are the same.
- g) $\{f_{\mathbf{s}\mathbf{t}} \mid \mathbf{s}, \mathbf{t} \in \mathcal{T}_n^{ud}(\lambda), (f, \lambda) \in \Lambda_{r,n}^+\}$ is an F -basis of $\mathcal{B}_{r,n}$. Further, we have $f_{\mathbf{s}\mathbf{t}} f_{\mathbf{u}\mathbf{v}} = \delta_{\mathbf{t}\mathbf{u}} \langle f_{\mathbf{t}}, f_{\mathbf{t}} \rangle f_{\mathbf{s}\mathbf{v}}$ where $\mathbf{s}, \mathbf{t}, \mathbf{u}, \mathbf{v}$ are updown tableaux and $\langle \cdot, \cdot \rangle$ is the invariant bilinear form defined on the cell module $\Delta(f, \lambda)$.

By Lemma 4.3(f), we can compute the Gram determinant associated to $\Delta(f, \lambda)$ by computing each $\langle f_{\mathbf{t}}, f_{\mathbf{t}} \rangle$, for $\mathbf{t} \in \mathcal{T}_n^{ud}(\lambda)$.

Given two $\mathbf{s}, \mathbf{t} \in \mathcal{T}_n^{ud}(\lambda)$ and a positive integer $k \leq n-1$. We write $\mathbf{s} \stackrel{k}{\sim} \mathbf{t}$ if $\mathbf{s}_j = \mathbf{t}_j$ for $1 \leq j \leq n$ and $j \neq k$.

Definition 4.4. For any $\mathbf{s}, \mathbf{t} \in \mathcal{T}_n^{ud}(\lambda)$ and a positive integer $k \leq n-1$, define $T_{\mathbf{t}\mathbf{s}}(k), E_{\mathbf{t}\mathbf{s}}(k) \in F$ by declaring that

$$f_{\mathbf{t}} T_k = \sum_{\mathbf{s} \in \mathcal{T}_n^{ud}(\lambda)} T_{\mathbf{t}\mathbf{s}}(k) f_{\mathbf{s}}, \quad f_{\mathbf{t}} E_k = \sum_{\mathbf{s} \in \mathcal{T}_n^{ud}(\lambda)} E_{\mathbf{t}\mathbf{s}}(k) f_{\mathbf{s}}.$$

Standard arguments prove the following result (cf. [18, 6.8–6.9]).

Lemma 4.5. Suppose $\mathbf{t} \in \mathcal{T}_n^{ud}(\lambda)$ and $(f, \lambda) \in \Lambda_{r,n}^+$.

- a) $\mathbf{s} \stackrel{k}{\sim} \mathbf{t}$ if either $T_{\mathbf{t}\mathbf{s}}(k) \neq 0$ or $E_{\mathbf{t}\mathbf{s}}(k) \neq 0$.
- b) $f_{\mathbf{t}} E_k = 0$ if $\mathbf{t}_{k-1} \neq \mathbf{t}_{k+1}$ for any $1 \leq k \leq n-1$.
- c) Assume $\mathbf{t}_{k-1} \neq \mathbf{t}_{k+1}$.
 - (i) If $\mathbf{t}_k \ominus \mathbf{t}_{k-1}$ and $\mathbf{t}_k \ominus \mathbf{t}_{k+1}$ are in the same row of a component, then $f_{\mathbf{t}} T_k = q f_{\mathbf{t}}$.
 - (ii) If $\mathbf{t}_k \ominus \mathbf{t}_{k-1}$ and $\mathbf{t}_k \ominus \mathbf{t}_{k+1}$ are in the same column of a component, then $f_{\mathbf{t}} T_k = -q^{-1} f_{\mathbf{t}}$.
- d) Assume $\mathbf{t}_{k-1} = \mathbf{t}_{k+1}$.
 - (i) $f_{\mathbf{t}} E_k = \sum_{\mathbf{s} \stackrel{k}{\sim} \mathbf{t}} E_{\mathbf{t}\mathbf{s}}(k) f_{\mathbf{s}}$. Furthermore, $\langle f_{\mathbf{s}}, f_{\mathbf{s}} \rangle E_{\mathbf{t}\mathbf{s}}(k) = \langle f_{\mathbf{t}}, f_{\mathbf{t}} \rangle E_{\mathbf{s}\mathbf{t}}(k)$.
 - (ii) $f_{\mathbf{t}} T_k = \sum_{\mathbf{s} \stackrel{k}{\sim} \mathbf{t}} T_{\mathbf{t}\mathbf{s}}(k) f_{\mathbf{s}}$. Furthermore, $T_{\mathbf{t}\mathbf{s}}(k) = \delta_{\frac{E_{\mathbf{t}\mathbf{s}}(k) - \delta_{\mathbf{t}\mathbf{s}}}{c_{\mathbf{t}}(k)c_{\mathbf{s}}(k) - 1}}$.

Lemma 4.6. Suppose that $\mathbf{t} \in \mathcal{T}_n^{ud}(\lambda)$ with $\mathbf{t}_{k-1} \neq \mathbf{t}_{k+1}$ and $\mathbf{t}\mathbf{s}_k \in \mathcal{T}_n^{ud}(\lambda)$. Then $f_{\mathbf{t}} T_k = T_{\mathbf{t},\mathbf{t}}(k) f_{\mathbf{t}} + T_{\mathbf{t},\mathbf{t}\mathbf{s}_k}(k) f_{\mathbf{t}\mathbf{s}_k}$, with $T_{\mathbf{t},\mathbf{t}}(k) = \frac{\delta_{c_{\mathbf{t}}(k+1)}}{c_{\mathbf{t}}(k+1) - c_{\mathbf{t}}(k)}$. Suppose one of the following conditions holds:

- (1) $\mathbf{t}_{k-1} \subset \mathbf{t}_k \subset \mathbf{t}_{k+1}$,
- (2) $\mathbf{t}_{k-1} \supset \mathbf{t}_k \subset \mathbf{t}_{k+1}$ such that $(\tilde{p}, l) > (p, i)$ where $\mathbf{t}_{k-1} \setminus \mathbf{t}_k = (p, i, \nu_i^{(p)})$, $\mathbf{t}_{k+1} \setminus \mathbf{t}_k = (\tilde{p}, \ell, \mu_{\ell}^{(\tilde{p})})$, $\mathbf{t}_{k-1} = \nu$ and $\mathbf{t}_{k+1} = \mu$.

Then

$$T_{\mathbf{t},\mathbf{t}\mathbf{s}_k}(k) = \begin{cases} 1 - \frac{c_{\mathbf{t}}(k)}{c_{\mathbf{t}}(k+1)} T_{\mathbf{t},\mathbf{t}}^2(k), & \text{if } \mathbf{t}\mathbf{s}_k \triangleright \mathbf{t}, \\ 1, & \text{if } \mathbf{t}\mathbf{s}_k \triangleleft \mathbf{t}. \end{cases}$$

Proof. By defining relation 2.1(f),

$$(4.7) \quad f_{\mathbf{t}} T_k X_k - f_{\mathbf{t}} X_{k+1} T_k = \delta f_{\mathbf{t}} X_{k+1} (E_k - 1).$$

Since we are assuming that $\mathbf{t}_{k-1} \neq \mathbf{t}_{k+1}$, $\mathbf{s} \in \{\mathbf{t}, \mathbf{t}\mathbf{s}_k\}$ if $\mathbf{s} \stackrel{k}{\sim} \mathbf{t}$. Comparing the coefficients of $f_{\mathbf{t}}$ on both sides of (4.7) and using Lemma 4.5(b) yields the formula for $T_{\mathbf{t},\mathbf{t}}(k)$, as required.

First, we assume that $\mathfrak{t} \triangleright \mathfrak{ts}_k$ and $\mathfrak{t}_{k-1} \subset \mathfrak{t}_k \subset \mathfrak{t}_{k+1}$. By Lemma 4.3(a),

$$f_{\mathfrak{t}} = \mathfrak{m}_{\mathfrak{t}} + \sum_{\mathfrak{u} \succ \mathfrak{t}} a_{\mathfrak{u}} f_{\mathfrak{u}}$$

for some scalars $a_{\mathfrak{u}} \in F$.

By Lemma 3.21(a) and Lemma 4.3(b), $\mathfrak{m}_{\mathfrak{t}} T_k = \mathfrak{m}_{\mathfrak{ts}_k} + \sum_{\mathfrak{v} \succ \mathfrak{ts}_k}^k b_{\mathfrak{v}} f_{\mathfrak{v}}$ for some scalars $b_{\mathfrak{v}} \in R$. We claim that $f_{\mathfrak{ts}_k}$ can not appear in the expressions of $f_{\mathfrak{u}} T_k$ with non-zero coefficient. Otherwise, $\mathfrak{u} \stackrel{k}{\sim} \mathfrak{ts}_k$, forcing $\mathfrak{u} \in \{\mathfrak{t}, \mathfrak{ts}_k\}$. This is a contradiction since $\mathfrak{ts}_k \triangleleft \mathfrak{t}$. By Lemma 4.3(b), the coefficient of $f_{\mathfrak{ts}_k}$ in $f_{\mathfrak{t}} T_k$ is 1.

Suppose that $\mathfrak{t}_{k-1} \supset \mathfrak{t}_k \subset \mathfrak{t}_{k+1}$. By Lemma 3.21(b),

$$\mathfrak{m}_{\mathfrak{t}} T_k^{-1} = \mathfrak{m}_{\mathfrak{ts}_k} + \sum_{\mathfrak{u} \succ \mathfrak{ts}_k}^k a_{\mathfrak{u}} \mathfrak{m}_{\mathfrak{u}},$$

for some scalars $a_{\mathfrak{u}} \in F$. Using 2.1(b) to rewrite the above equality yields

$$\mathfrak{m}_{\mathfrak{t}} T_k = \mathfrak{m}_{\mathfrak{ts}_k} + \sum_{\mathfrak{u} \succ \mathfrak{ts}_k}^k a_{\mathfrak{u}} \mathfrak{m}_{\mathfrak{u}} + \delta \mathfrak{m}_{\mathfrak{t}} - \delta \mathfrak{m}_{\mathfrak{t}} E_k.$$

We use Lemma 4.3(b) to write the terms on the right hand side of the above equality as a linear combination of orthogonal basis elements. Since $\mathfrak{ts}_k \triangleleft \mathfrak{t}$, $f_{\mathfrak{ts}_k}$ can not appear in the expression of $\sum_{\mathfrak{u} \succ \mathfrak{ts}_k}^k a_{\mathfrak{u}} \mathfrak{m}_{\mathfrak{u}} + \delta \mathfrak{m}_{\mathfrak{t}}$.

We claim that $f_{\mathfrak{ts}_k}$ can not appear in the expression of $\mathfrak{m}_{\mathfrak{t}} E_k$. Otherwise, by Lemma 4.3(b), we write $\mathfrak{m}_{\mathfrak{t}} = \sum_{\mathfrak{v} \succeq \mathfrak{t}} a_{\mathfrak{v}} f_{\mathfrak{v}}$. Therefore, there is a \mathfrak{v} such that $f_{\mathfrak{ts}_k}$ appears in the expression of $f_{\mathfrak{v}} E_k$ with non-zero coefficient. So, $\mathfrak{v} \stackrel{k}{\sim} \mathfrak{ts}_k$, forcing $\mathfrak{v}_{k-1} \neq \mathfrak{v}_{k+1}$. Thus $f_{\mathfrak{v}} E_k = 0$, a contradiction. This completes the proof of our claim. Therefore, the coefficient of $f_{\mathfrak{ts}_k}$ in $\mathfrak{m}_{\mathfrak{t}} T_k$ is 1.

Using Lemma 4.3(b) again, we write $\mathfrak{m}_{\mathfrak{t}} = f_{\mathfrak{t}} + \sum_{\mathfrak{u} \succ \mathfrak{t}} a_{\mathfrak{u}} f_{\mathfrak{u}}$ for some scalars $a_{\mathfrak{u}} \in F$. If $f_{\mathfrak{ts}_k}$ appears in the expression of $\sum_{\mathfrak{u} \succ \mathfrak{t}} a_{\mathfrak{u}} f_{\mathfrak{u}} T_k$, then $f_{\mathfrak{ts}_k}$ must appear in the expression of $f_{\mathfrak{u}} T_k$ for some \mathfrak{u} . So, $\mathfrak{ts}_k \stackrel{k}{\sim} \mathfrak{u}$, forcing $\mathfrak{u} \in \{\mathfrak{t}, \mathfrak{ts}_k\}$. This contradicts the fact $\mathfrak{u} \succ \mathfrak{t}$. So, the coefficient of $f_{\mathfrak{ts}_k}$ in $f_{\mathfrak{t}} T_k$ is 1.

We have proved that

$$(4.8) \quad f_{\mathfrak{t}} T_k = \frac{\delta c_{\mathfrak{t}}(k+1)}{c_{\mathfrak{t}}(k+1) - c_{\mathfrak{t}}(k)} f_{\mathfrak{t}} + f_{\mathfrak{ts}_k},$$

if $\mathfrak{ts}_k \triangleleft \mathfrak{t}$ and one of conditions (1)-(2) holds. Multiplying T_k on both sides of (4.8) and using 2.1(b) yields

$$(4.9) \quad f_{\mathfrak{ts}_k} T_k = f_{\mathfrak{t}} + \delta f_{\mathfrak{t}} T_k - \frac{\delta c_{\mathfrak{t}}(k+1)}{c_{\mathfrak{t}}(k+1) - c_{\mathfrak{t}}(k)} f_{\mathfrak{t}} T_k - \delta \rho f_{\mathfrak{t}} E_k.$$

Note that $\mathfrak{t}_{k-1} \neq \mathfrak{t}_{k+1}$. By Lemma 4.5(b), $f_{\mathfrak{ts}_k} E_k = 0$. Using (4.8) to simplify (4.9) and switching the role between \mathfrak{ts}_k and \mathfrak{t} yields the formula for $T_{\mathfrak{t}, \mathfrak{ts}_k}(k)$ provided $\mathfrak{ts}_k \triangleright \mathfrak{t}$ together with one of conditions in (1)-(2) being true. \square

Note that $\langle f_{\mathfrak{t}} T_k, f_{\mathfrak{ts}_k} \rangle = \langle f_{\mathfrak{t}}, f_{\mathfrak{ts}_k} T_k \rangle$. By Lemma 4.6, we have the following result immediately.

Corollary 4.10. *Suppose $\mathfrak{t} \in \mathcal{T}_n^{ud}(\lambda)$ with $(f, \lambda) \in \Lambda_{r,n}^+$ and $\mathfrak{t}_{k-1} \neq \mathfrak{t}_{k+1}$. If $\mathfrak{ts}_k \in \mathcal{T}_n^{ud}(\lambda)$, $\mathfrak{ts}_k \triangleleft \mathfrak{t}$ and one of the conditions (1)-(2) in Lemma 4.6 holds, then*

$$\langle f_{\mathfrak{ts}_k}, f_{\mathfrak{ts}_k} \rangle = \left(1 - \frac{\delta^2 c_{\mathfrak{t}}(k) c_{\mathfrak{t}}(k+1)}{(c_{\mathfrak{t}}(k+1) - c_{\mathfrak{t}}(k))^2}\right) \langle f_{\mathfrak{t}}, f_{\mathfrak{t}} \rangle.$$

Let a be an integer. Let $[a]_{q^2} = \frac{q^{2a}-1}{q^2-1}$. For any partition $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_k)$, let $[\lambda]_{q^2}! = [\lambda_1]_{q^2}! [\lambda_2]_{q^2}! \cdots [\lambda_k]_{q^2}!$. If $\lambda = (\lambda^{(1)}, \lambda^{(2)}, \dots, \lambda^{(r)}) \in \Lambda_r^+(n)$, let $[\lambda]_{q^2}! = [\lambda^{(1)}]_{q^2}! [\lambda^{(2)}]_{q^2}! \cdots [\lambda^{(r)}]_{q^2}!$.

Lemma 4.11. (cf. [18, 6.11]) Suppose that $(f, \lambda) \in \Lambda_{r,n_1}^+$ and $(f, \mu) \in \Lambda_{r,n_2}^+$. Let $[\lambda] = [a_1, a_2, \dots, a_r]$ and $[\mu] = [b_1, b_2, \dots, b_r]$. Then

$$(4.12) \quad \frac{\langle f_{t^\lambda}, f_{t^\lambda} \rangle}{\langle f_{t^\mu}, f_{t^\mu} \rangle} = \frac{[\lambda]_{q^2}! \prod_{j=2}^r \prod_{k=1}^{a_{j-1}} (c_{t^\lambda}(k) - u_j)}{[\mu]_{q^2}! \prod_{j=2}^r \prod_{k=1}^{b_{j-1}} (c_{t^\mu}(k) - u_j)}.$$

Proof. This can be verified by arguments in the proof of [18, 6.11]. We leave the details to the reader. \square

4.13. Suppose that $\lambda \in \Lambda_r^+(n - 2f)$. Following [18], we define $\mathcal{A}(\lambda)$ (resp. $\mathcal{R}(\lambda)$) to be the set of all addable (resp. removable) nodes of λ . Given a removable (resp. an addable) node $p = (s, k, \lambda_k)$ (resp. $(s, k, \lambda_k + 1)$) of λ , define

- a) $\mathcal{R}(\lambda)^{<p} = \{(h, l, \lambda_l) \in \mathcal{R}(\lambda) \mid (h, l) > (s, k)\}$,
- b) $\mathcal{A}(\lambda)^{<p} = \{(h, l, \lambda_l + 1) \in \mathcal{A}(\lambda) \mid (h, l) > (s, k)\}$,
- c) $\mathcal{A}\mathcal{R}(\lambda)^{\geq p} = \{(h, l, \lambda_l) \in \mathcal{R}(\lambda) \mid (h, l) \leq (s, k)\} \cup \{(h, l, \lambda_l + 1) \in \mathcal{A}(\lambda) \mid (h, l) \leq (s, k)\}$.

Following [16], let $\hat{\mathbf{t}} = (t_0, t_1, t_2, \dots, t_{n-1})$ and $\tilde{\mathbf{t}} = (s_0, s_1, s_2, \dots, s_{n-1}, t_n)$ with $t_{n-1} = \mu$ and $(s_0, s_1, s_2, \dots, s_{n-1}) = \mathbf{t}^\mu$ for any $\mathbf{t} = (t_0, t_1, t_2, \dots, t_n) \in \mathcal{T}_n^{ud}(\lambda)$. Standard arguments prove the following result (cf. [18, 6.15]).

Proposition 4.14. Assume that $\mathbf{t} \in \mathcal{T}_n^{ud}(\lambda)$ with $(f, \lambda) \in \Lambda_{r,n}^+$. If $t_{n-1} = \mu$, then

$$\langle f_{\mathbf{t}}, f_{\mathbf{t}} \rangle = \langle f_{\tilde{\mathbf{t}}}, f_{\tilde{\mathbf{t}}} \rangle \frac{\langle f_{\tilde{\mathbf{t}}}, f_{\tilde{\mathbf{t}}} \rangle}{\langle f_{\mathbf{t}^\mu}, f_{\mathbf{t}^\mu} \rangle}.$$

By Proposition 4.14, we can compute $\langle f_{\mathbf{t}}, f_{\mathbf{t}} \rangle$ recursively if we know how to compute $\frac{\langle f_{\tilde{\mathbf{t}}}, f_{\tilde{\mathbf{t}}} \rangle}{\langle f_{\mathbf{t}^\mu}, f_{\mathbf{t}^\mu} \rangle}$. There are three cases which will be given in Propositions 4.15, 4.18 and 4.24.

Proposition 4.15. Suppose that $\mathbf{t} \in \mathcal{T}_n^{ud}(\lambda)$ with $(f, \lambda) \in \Lambda_{r,n}^+$. If $\hat{\mathbf{t}} = \mathbf{t}^\mu$ with $t_n = t_{n-1} \cup \{p\}$ and $p = (m, k, \lambda_k^{(m)})$, then

$$(4.16) \quad \frac{\langle f_{\mathbf{t}}, f_{\mathbf{t}} \rangle}{\langle f_{\mathbf{t}^\mu}, f_{\mathbf{t}^\mu} \rangle} = \frac{(-1)^{r-m} q^{2k}}{u_m(1-q^2)} \frac{\prod_{a \in \mathcal{A}(\lambda)^{<p}} (c_\lambda(a) - c_\lambda(p)^{-1})}{\prod_{a \in \mathcal{R}(\lambda)^{<p}} (c_\lambda(a)^{-1} - c_\lambda(p)^{-1})}.$$

Proof. Let $\lambda = [a_1, a_2, \dots, a_r]$, and $\mathbf{t} = t^\lambda s_{a,n}$ where $a = 2f + a_{m-1} + \sum_{i=1}^k \lambda_i^{(m)}$. Note that $\mathbf{t} \triangleleft t s_{n-1} \triangleleft \cdots \triangleleft t s_{n,a} = t^\lambda$, and $\mathbf{t}_a \subset \mathbf{t}_{a+1} \subset \cdots \subset \mathbf{t}_n$. Applying Corollary 4.10 on the pairs $\{f_{t^\lambda s_{a,j}}, f_{t^\lambda s_{a,j+1}}\}$, $a \leq j \leq n-1$, we have

$$(4.17) \quad \langle f_{\mathbf{t}}, f_{\mathbf{t}} \rangle = \langle f_{t^\lambda}, f_{t^\lambda} \rangle \prod_{j=a+1}^n (1 - \delta^2 \frac{c_{t^\lambda}(j) c_{\mathbf{t}}(a)}{(c_{t^\lambda}(j) - c_{\mathbf{t}}(a))^2}).$$

Simplifying (4.17) via the definition of $c_{\mathbf{t}}(j)$ $a \leq j \leq n$ together with (4.12) yields (4.16). \square

Proposition 4.18. Suppose that $\mathbf{t} \in \mathcal{T}_n^{ud}(\lambda)$ with $\lambda \in \Lambda_r^+(n - 2f)$ and $\mathbf{t}^\mu = \hat{\mathbf{t}}$. If $t_{n-1} = t_n \cup \{p\}$ with $p = (s, k, \mu_k^{(s)})$ such that $\mu^{(j)} = \emptyset$ for all integers $j, s < j \leq r$ and $l(\mu^{(s)}) = k$, then

$$(4.19) \quad \frac{\langle f_{\mathbf{t}}, f_{\mathbf{t}} \rangle}{\langle f_{\mathbf{t}^\mu}, f_{\mathbf{t}^\mu} \rangle} = [\mu_k^{(s)}]_{q^2} E_{\mathbf{t}\mathbf{t}}(n-1) \prod_{j=s+1}^r (u_s q^{2(\mu_k^{(s)} - k)} - u_j)$$

Proof. We have

$$\begin{aligned}
& E^{f,n} T_{n-1,n-2f+1} X_{n-2f+1}^k T_{n-2f+1,n-1} F_{t,n} F_{t,n-1} E_{n-1} \\
& \stackrel{2.1h}{=} E_{n-1} \prod_{i=2}^f E_{n-2i+1,n-2i+3} X_{n-2f+1}^k T_{n-2f+1,n-1} F_{t,n} F_{t,n-1} E_{n-1} \\
& \stackrel{2.1j}{=} E_{n-1} X_{n-1}^k \prod_{i=2}^f E_{n-2i+1,n-2i+3} T_{n-2f+1,n-1} F_{t,n} F_{t,n-1} E_{n-1} \\
& \stackrel{2.1h,i}{=} E^{f,n} X_n^{-k} F_{t,n} F_{t,n-1} E_{n-1}.
\end{aligned}$$

and

$$\begin{aligned}
& E^{f,n} T_{n-1,n-2f+1} X_{n-2f+1}^k T_{n-2f,n-1} F_{t,n} F_{t,n-1} E_{n-1} \\
& \stackrel{2.1h}{=} E_{n-1} \prod_{i=2}^f E_{n-2i+1,n-2i+3} X_{n-2f+1}^k T_{n-2f,n-1} F_{t,n} F_{t,n-1} E_{n-1} \\
& \stackrel{2.1j}{=} E_{n-1} X_n^{-k} \prod_{i=f}^2 E_{n-2i+2,n-2i} T_{n-2f,n-1} F_{t,n} F_{t,n-1} E_{n-1} \\
& \stackrel{2.1b,c}{=} E_{n-1} \prod_{i=f}^2 E_{n-2i+2,n-2i} T_{n-2f,n-1} X_n^{-k} F_{t,n} F_{t,n-1} E_{n-1} \\
& \stackrel{2.1h}{=} E^{f,n} T_{n-1,n-2f+1} T_{n-2f,n-1} X_n^{-k} F_{t,n} F_{t,n-1} E_{n-1} \\
& \stackrel{2.1c}{=} E^{f,n} T_{n-2f,n-2} T_{n-1,n-2f} X_n^{-k} F_{t,n} F_{t,n-1} E_{n-1} \\
& \stackrel{2.1b,c}{=} E^{f-1,n-2} T_{n-2f,n-2} E_{n-1} T_{n-2} X_n^{-k} F_{t,n} F_{t,n-1} E_{n-1} T_{n-2,n-2f}
\end{aligned}$$

By [19, 4.27a] and Definition 2.1j, we can write $E_{n-1} T_{n-2} X_n^{-k} F_{t,n} F_{t,n-1} E_{n-1}$ as an R -linear combination of elements $E_{n-1} g(X_1^\pm, \dots, X_{n-2}^\pm) X_{n-2}^\ell$ where $g(X_1^\pm, \dots, X_{n-2}^\pm)$ is a polynomial in variables $X_1^\pm, \dots, X_{n-2}^\pm$, which is in the center of $\mathcal{B}_{r,n-2}$. Therefore,

$$\begin{aligned}
& E^{f-1,n-2} T_{n-2f,n-2} E_{n-1} X_{n-2}^\ell g(X_1^\pm, \dots, X_{n-2}^\pm) T_{n-2,n-2f} \\
& = E^{f,n} T_{n-2f,n-2} X_{n-2}^\ell T_{n-2,n-2f} g(X_1^\pm, \dots, X_{n-2}^\pm) \\
& = X_{n-2f}^\ell E_{n-1} \prod_{i=f}^2 E_{n-2i+2,n-2i} T_{n-2,n-2f} g(X_1^\pm, \dots, X_{n-2}^\pm) \\
& = E^{f,n} X_{n-2f}^\ell g(X_1^\pm, \dots, X_{n-2}^\pm).
\end{aligned}$$

Note that $f_t = \mathbf{m}_t F_t$. Here we use \mathbf{m}_t instead of $\mathbf{m}_t \pmod{\mathcal{B}_{r,n}^{\triangleright(f,\lambda)}}$. By (3.12), $f_t E_{n-1} = E_{n-1} T_{n-1,n-2f+1} \mathbf{m}_\mu T_{n-2f+1,n-1} b_{t,n-2} F_t E_{n-1}$

$$\begin{aligned}
& = M_\lambda E^{f,n} T_{n-1,n-2f+1} \prod_{j=s+1}^r (X_{n-2f+1} - u_j) \\
& \quad \times \sum_{i=a_{s,k-1}+1}^{n-2f+1} q^{n-2f+1-i} T_{n-2f+1,i} T_{n-2f+1,n-1} F_{t,n} F_{t,n-1} E_{n-1} b_{t,n-2} \prod_{k=1}^{n-2} F_{t,k} \\
& = M_\lambda E^{f,n} T_{n-1,n-2f+1} \prod_{j=s+1}^r (X_{n-2f+1} - u_j) \\
& \quad (1 + T_{n-2f} \sum_{i=a_{s,k-1}+1}^{n-2f} q^{n-2f+1-i} T_{n-2f,i}) T_{n-2f+1,n-1} F_{t,n} F_{t,n-1} E_{n-1} b_{t,n-2} \prod_{k=1}^{n-2} F_{t,k}.
\end{aligned}$$

By [19, 4.21] and our two equalities in the beginning of the proof, we can find $\Phi_t, \Psi_\ell \in F[X_1^\pm, X_2^\pm, \dots, X_{n-2}^\pm] \cap Z(\mathcal{B}_{r,n-2})$, $\ell \in \mathbb{Z}$ such that

$$f_t E_{n-1} = E^{f,n} M_\lambda (\Phi_t + \sum_{\ell} X_{n-2f}^\ell \Psi_\ell \sum_{i=a_{s,k-1}+1}^{n-2f} q^{n-2f+1-i} T_{n-2f,i}) b_{t_{n-2}} \prod_{k=1}^{n-2} F_{t,k}.$$

More explicitly, Φ_t is defined by (4.20) as follows:

$$(4.20) \quad E_{n-1} \prod_{j=s+1}^r (X_n^{-1} - u_j) F_{t,n-1} F_{t,n} E_{n-1} = \Phi_t E_{n-1}.$$

Now, we use [19, 5.8] and [13, 3.7] to get

$$f_t E_{n-1} = E^{f,n} M_\lambda b_{t_{n-2}} (\Phi_t + q[\lambda_k^{(s)}]_{q^2} \sum_{\ell} \Psi_\ell c_{t^\lambda}(n-2f)^\ell) \prod_{k=1}^{n-2} F_{t,k}.$$

Let $u \in \mathcal{T}_n^{ud}(\lambda)$ such that u is minimal in the sense of $u \stackrel{n-1}{\sim} t$. Then $m_u = E^{f,n} M_\lambda b_{t_{n-2}} \pmod{\mathcal{B}_{r,n}^{\triangleright(f,\lambda)}}$. Therefore, $m_u X_k = c_{t^\lambda}(k) m_u$ for all $1 \leq k \leq n-2$. We have

$$f_t E_{n-1} = (\Phi_{t,\lambda} + q[\lambda_k^{(s)}]_{q^2} \Psi_{t,\lambda}) m_u$$

where $\Psi_{t,\lambda} = \sum_{\ell} \Psi_\ell c_{t^\lambda}(n-2f)^\ell$ and $\Phi_{t,\lambda}$ and $\Psi_{t,\lambda}$ are obtained from Φ_t and Ψ_ℓ by using $c_{t^\lambda}(k)$ instead of X_k , $1 \leq k \leq n-2$. By Lemma 4.3(b) and Definition 4.4,

$$(4.21) \quad E_{tu}(n-1) = \Phi_{t,\lambda} + q[\lambda_k^{(s)}]_{q^2} \Psi_{t,\lambda}.$$

We compute $\Phi_{t,\lambda}$ and $\Psi_{t,\lambda}$ as follows. By (4.20),

$$\begin{aligned} \Phi_{t,\lambda} f_t E_{n-1} &= f_t E_{n-1} \prod_{j=s+1}^r (X_n^{-1} - u_j) F_{t,n-1} F_{t,n} E_{n-1} \\ &= E_{tt}(n-1) \prod_{j=s+1}^r (c_t^{-1}(n) - u_j) f_t E_{n-1}. \end{aligned}$$

When we get the last equation, we use the fact that $f_s F_{t,n-1} F_{t,n} = 0$ for all $s \in \mathcal{T}_n^{ud}(\lambda)$ with $s \stackrel{n-1}{\sim} t$ and $s \neq t$, which follows from Lemma 4.3(d). So,

$$(4.22) \quad \Phi_{t,\lambda} = E_{tt}(n-1) \prod_{j=s+1}^r (c_t^{-1}(n) - u_j).$$

Similarly, we can verify

$$(4.23) \quad \Psi_{t,\lambda} = q E_{tt}(n-1) \prod_{j=s+1}^r (c_t^{-1}(n) - u_j).$$

By (4.22)–(4.23),

$$E_{tu}(n-1) = (1 + q^2[\lambda_k^{(s)}]_{q^2}) E_{tt}(n-1) \prod_{j=s+1}^r (c_t^{-1}(n) - u_j).$$

On the other hand, by similar arguments for $f_{t^\lambda u} f_{u t^\lambda}$ in [18, 6.22] for cyclotomic Nazarov-Wenzl algebra, we can verify

$$f_{t^\lambda u} f_{u t^\lambda} \equiv E_{uu}(n-1) \langle f_v, f_v \rangle f_{t^\lambda t^\lambda} \pmod{\mathcal{B}_{r,n}^{\triangleright(f,\lambda)}},$$

where $v = (u_1, u_2, \dots, u_{n-2}) \in \mathcal{T}_{n-2}^{ud}(\lambda)$. So, $\langle f_u, f_u \rangle = E_{uu}(n-1) \langle f_v, f_v \rangle$. Note that

In [19, 4.7], Rui and Xu introduced rational functions $W_k(y, \mathfrak{s})$ in variable y for any $\mathfrak{s} \in \mathcal{T}_n^{ud}(\lambda)$ such that

$$f_{\mathfrak{s}} E_k \frac{y}{y - X_k} E_k = E_k W_k(y, \mathfrak{s}).$$

Suppose that $\mathfrak{s} = \mathfrak{t}$. By comparing the coefficient of $f_{\mathfrak{u}}$ on both sides of the above equality, we have

$$E_{\mathfrak{tu}}(n-1)E_{\mathfrak{ut}}(n-1) = E_{\mathfrak{tt}}(n-1)E_{\mathfrak{uu}}(n-1).$$

Note that $[\mu_k^{(s)}]_{q^2} = 1 + q^2[\lambda_k^{(s)}]_{q^2}$ and $c_{\mathfrak{t}}(n) = u_s^{-1}q^{2(k-\mu_k^{(s)})}$. Therefore,

$$\begin{aligned} \frac{\langle f_{\mathfrak{t}}, f_{\mathfrak{t}} \rangle}{\langle f_{\mathfrak{t}^\mu}, f_{\mathfrak{t}^\mu} \rangle} &= \frac{E_{\mathfrak{tu}}(n-1)\langle f_{\mathfrak{u}}, f_{\mathfrak{u}} \rangle}{E_{\mathfrak{ut}}(n-1)\langle f_{\mathfrak{t}^\mu}, f_{\mathfrak{t}^\mu} \rangle} \\ &= \frac{E_{\mathfrak{tu}}^2(n-1)\langle f_{\mathfrak{u}}, f_{\mathfrak{u}} \rangle}{E_{\mathfrak{uu}}(n-1)E_{\mathfrak{tt}}(n-1)\langle f_{\mathfrak{t}^\mu}, f_{\mathfrak{t}^\mu} \rangle} = \frac{E_{\mathfrak{tu}}^2(n-1)\langle f_{\mathfrak{v}}, f_{\mathfrak{v}} \rangle}{E_{\mathfrak{tt}}(n-1)\langle f_{\mathfrak{t}^\mu}, f_{\mathfrak{t}^\mu} \rangle} \\ &= (1 + q^2[\lambda_k^{(s)}]_{q^2})^2 E_{\mathfrak{tt}}(n-1) \prod_{j=s+1}^r (c_{\mathfrak{t}}^{-1}(n) - u_j)^2 \frac{\langle f_{\mathfrak{v}}, f_{\mathfrak{v}} \rangle}{\langle f_{\mathfrak{t}^\mu}, f_{\mathfrak{t}^\mu} \rangle} \end{aligned}$$

By Lemma 4.11, $\frac{\langle f_{\mathfrak{v}}, f_{\mathfrak{v}} \rangle}{\langle f_{\mathfrak{t}^\mu}, f_{\mathfrak{t}^\mu} \rangle} = [\mu_k^{(s)}]_{q^2}^{-1} \prod_{j=s+1}^r (u_s q^{2(\mu_k^{(s)} - k)} - u_j)^{-1}$. So,

$$\frac{\langle f_{\mathfrak{t}}, f_{\mathfrak{t}} \rangle}{\langle f_{\mathfrak{t}^\mu}, f_{\mathfrak{t}^\mu} \rangle} = [\mu_k^{(s)}]_{q^2} E_{\mathfrak{tt}}(n-1) \prod_{j=s+1}^r (u_s q^{2(\mu_k^{(s)} - k)} - u_j).$$

□

Proposition 4.24. Suppose that $\lambda = (\lambda^{(1)}, \lambda^{(2)}, \dots, \lambda^{(s)}, \emptyset, \dots, \emptyset) \in \Lambda_r^+(n-2f)$ and $l(\lambda^{(s)}) = l$. Let $\mathfrak{t} \in \mathcal{T}_n^{ud}(\lambda)$ with $(f, \lambda) \in \Lambda_{r,n}^+$ such that $\hat{\mathfrak{t}} = \mathfrak{t}^\mu$, and $\mathfrak{t}_{n-1} = \mathfrak{t}_n \cup \{p\}$ with $p = (m, k, \mu_k^{(m)})$ and $(m, k) < (s, l)$. Let $\mu = [b_1, b_2, \dots, b_r]$. We define $\mathfrak{u} = \mathfrak{t}_{s, a+1}$ with $a = 2(f-1) + b_{m-1} + \sum_{j=1}^k \mu_j^{(m)}$ and $\mathfrak{v} = (\mathfrak{u}_1, \dots, \mathfrak{u}_{a+1})$. Then

$$(4.25) \quad \frac{\langle f_{\mathfrak{t}}, f_{\mathfrak{t}} \rangle}{\langle f_{\mathfrak{t}^\mu}, f_{\mathfrak{t}^\mu} \rangle} = [\mu_k^{(m)}]_{q^2} E_{\mathfrak{vv}}(a) (u_m q^{-2k} - u_m^{-1} q^{-2(\mu_k^{(m)} - k)})^{-1} A$$

where $A = \prod_{j=m+1}^r \frac{(u_m q^{2(\mu_k^{(m)} - k)} - u_j)}{(u_j - u_m^{-1} q^{-2(\mu_k^{(m)} - k)})} \frac{\prod_{a \in \mathcal{A}(\mu) < p} (c_\mu(a) - c_\mu(p))}{\prod_{b \in \mathcal{B}(\mu) < p} (c_\mu(b)^{-1} - c_\mu(p))}.$

Proof. We have $\mathfrak{t} \triangleleft \mathfrak{t}_{n-1} \triangleleft \dots \triangleleft \mathfrak{t}_{s, a+1} = \mathfrak{u}$, and $\mathfrak{v} = (\mathfrak{u}_1, \mathfrak{u}_2, \dots, \mathfrak{u}_{a+1})$. Using Corollary 4.10 repeatedly yields

$$(4.26) \quad \langle f_{\mathfrak{t}}, f_{\mathfrak{t}} \rangle = \langle f_{\mathfrak{u}}, f_{\mathfrak{u}} \rangle \prod_{j=a+2}^n (1 - \delta^2 \frac{c_{\mathfrak{u}}(j)c_{\mathfrak{u}}(a+1)}{(c_{\mathfrak{u}}(j) - c_{\mathfrak{u}}(a+1))^2}).$$

By Propositions 4.15 and 4.18, we have

$$(4.27) \quad \frac{\langle f_{\mathfrak{u}}, f_{\mathfrak{u}} \rangle}{\langle f_{\mathfrak{t}^\mu}, f_{\mathfrak{t}^\mu} \rangle} = E_{\mathfrak{vv}}(a) [\mu_k^{(m)}]_{q^2} \prod_{j=m+1}^r (u_m q^{2(\mu_k^{(m)} - k)} - u_j).$$

Simplifying (4.26) via the definition of $c_{\mathfrak{u}}(j)$, $a+1 \leq j \leq n$ together with (4.27) yields (4.25), as required. □

Assume that $(f, \lambda) \in \Lambda_{r,n}^+$ and $(l, \mu) \in \Lambda_{r, n-1}^+$. Write $(l, \mu) \rightarrow (f, \lambda)$ if either $l = f$ and μ is obtained from λ by removing a removable node or $l = f-1$ and μ is obtained from λ by adding an addable node. Assume that $\mathcal{B}_{r,n}$ is semisimple. By Theorem 3.11,

$$(4.28) \quad \Delta(f, \lambda) \downarrow \cong \bigoplus_{(l, \mu) \rightarrow (f, \lambda)} \Delta(l, \mu),$$

where $\Delta(f, \lambda) \downarrow$ is $\Delta(f, \lambda)$ considered as $\mathcal{B}_{r,n-1}$ -module. We remark that (4.28) has been proved in [12] over \mathbb{C} .

Motivated by [16], we define $\gamma_{\lambda/\mu} \in F$ to be the scalar given by

$$(4.29) \quad \gamma_{\lambda/\mu} = \frac{\langle f_{\mathbf{t}}, f_{\mathbf{t}} \rangle}{\langle f_{\mathbf{t}^\mu}, f_{\mathbf{t}^\mu} \rangle}$$

where $\mathbf{t} \in \mathcal{T}_n^{ud}(\lambda)$ with $\hat{\mathbf{t}} = \mathbf{t}^\mu \in \mathcal{T}_{n-1}^{ud}(\mu)$. By [4, 5.1],

$$(4.30) \quad \text{rank} \Delta(f, \lambda) = \frac{r^f n! (2f-1)!!}{(2f)! \prod_{i=1}^r (a_i - a_{i-1})!} \prod_{i=1}^r \frac{a_i!}{\prod_{(k,\ell) \in \lambda^{(i)}} h_{k,\ell}^{\lambda^{(i)}}},$$

where $(f, \lambda) \in \Lambda_{r,n}^+$ and $[\lambda] = [a_1, a_2, \dots, a_r]$ and $h_{k,\ell}^{\lambda^{(i)}} = \lambda_k^{(i)} + \lambda_\ell^{(i)'} - k - \ell + 1$ is the hook length of (k, ℓ) in $\lambda^{(i)}$.

Standard arguments prove the following result (cf. [18, 6.38]).

Theorem 4.31. *Let $\mathcal{B}_{r,n}$ be over R where $R = \mathbb{Z}[u_1^\pm, \dots, u_r^\pm, q^\pm, \delta^{-1}]$ satisfying the assumption 2.2. Let $\det G_{f,\lambda}$ be the Gram determinant associated to the cell module $\Delta(f, \lambda)$ of $\mathcal{B}_{r,n}$. Then*

$$(4.32) \quad \det G_{f,\lambda} = \prod_{(l,\mu) \rightarrow (f,\lambda)} \det G_{l,\mu} \cdot \gamma_{\lambda/\mu}^{\text{rank} \Delta(l,\mu)} \in R.$$

Furthermore, $\text{rank} \Delta(l, \mu)$ is given by 4.30 and each scalar $\gamma_{\lambda/\mu}$ can be computed explicitly by Proposition 4.15, Proposition 4.18 and Proposition 4.24.

We compute $E_{\mathbf{s}\mathbf{s}}(k)$ for any $\mathbf{s} \in \mathcal{T}_n^{ud}(\lambda)$ and $1 \leq k \leq n$. In section 4 of [19], Rui and Xu have constructed the seminormal representations $\Delta(\lambda)$ for $\mathcal{B}_{r,n}$ where $\lambda \in \Lambda_r^+(n-2f)$. More explicitly, $\Delta(\lambda)$ has a basis $v_{\mathbf{s}}$, $\mathbf{s} \in \mathcal{T}_n^{ud}(\lambda)$. By standard arguments (cf. [16, 3.16]), one can verify that $f_{\mathbf{s}}$ constructed in the current section is equal to $v_{\mathbf{s}}$ up to a scalar. Therefore, $E_{\mathbf{s}\mathbf{s}}(k)$ can be computed by [19, 4.12-4.13]. We list such formulae as follows. Let $\varepsilon \in \{-1, 1\}$.

If r is odd and $\varrho^{-1} = \varepsilon \prod_{i=1}^r u_i$, then

$$(4.33) \quad E_{\mathbf{s}\mathbf{s}}(k) = \frac{1}{\varrho c_{\mathbf{s}}(k)} \left(\frac{c_{\mathbf{s}}(k) - c_{\mathbf{s}}(k)^{-1}}{\delta} + \varepsilon \right) \prod_{\alpha} \frac{c_{\mathbf{s}}(k) - c(\alpha)^{-1}}{c_{\mathbf{s}}(k) - c(\alpha)},$$

where α run over all addable and removable nodes of \mathbf{s}_{k-1} with $\alpha \neq \mathbf{s}_k \setminus \mathbf{s}_{k-1}$.

If r is even and $\varrho^{-1} = -\varepsilon q^\varepsilon \prod_{i=1}^r u_i$ then

$$(4.34) \quad E_{\mathbf{s}\mathbf{s}}(k) = \frac{1}{\varrho \delta} \left(1 - \frac{q^{-2\varepsilon}}{c_{\mathbf{s}}(k)^2} \right) \prod_{\alpha} \frac{c_{\mathbf{s}}(k) - c(\alpha)^{-1}}{c_{\mathbf{s}}(k) - c(\alpha)},$$

where α run over all addable and removable nodes of \mathbf{s}_{k-1} with $\alpha \neq \mathbf{s}_k \setminus \mathbf{s}_{k-1}$.

By Propositions 4.14, 4.15, 4.18 and 4.24 together with (4.33)-(4.34), we have the following result immediately.

Corollary 4.35. *Suppose that $(f, \lambda) \in \Lambda_{r,n}^+$. Let $[\lambda] = [a_1, a_2, \dots, a_r]$ and $\varepsilon \in \{-1, 1\}$. Then*

$$\langle f_{\mathbf{t}^\lambda}, f_{\mathbf{t}^\lambda} \rangle = \frac{[\lambda]!}{\varrho^f \delta^f} A \prod_{j=2}^r \prod_{k=1}^{a_{j-1}} (c_{\mathbf{t}^\lambda}(k) - u_j) \prod_{j=2}^r (u_1 - u_j)^f (u_1 - u_j^{-1})^f,$$

where

$$A = \begin{cases} (u_1^{-1} + q^{-\varepsilon})^f (-u_1^{-1} + q^\varepsilon)^f, & \text{if } 2 \nmid r \text{ and } \varrho^{-1} = \varepsilon \prod_{i=1}^r u_i, \\ (u_1 + q^\varepsilon)^f (u_1 - q^\varepsilon)^f u_1^{-2f}, & \text{if } 2 \mid r \text{ and } \varrho^{-1} = \varepsilon q^{-\varepsilon} \prod_{i=1}^r u_i. \end{cases}$$

Given an multi-partition of λ . We denote μ by $\lambda \cup p$ (resp. λ/p) if $Y(\mu)$ is obtained from λ by adding (resp. removing) the addable (resp. removable) node p . Let $p = (i, j, k)$ be the node which is in the j th-row, k th column of i th component of $Y(\lambda)$. We define $p^+ = (i, j, k+1)$ and $p^- = (i, j+1, k)$.

In the remainder of this section, we assume that

$$R = \mathbb{Z}[u_1^\pm, u_2^\pm, \dots, u_r^\pm, q^{\pm 1}, \delta^{-1}]$$

such that the assumption 2.2 holds. Let R_1 be the multiplicative sub-semigroup of R generated by $1, u_i^\pm, q^\pm, \delta^\pm$ and $u_i u_j^{-1} - q^{2d}$ for integers i, j, d with $|d| < n$ and $1 \leq i, j \leq r$. Let F_1 be the field of fraction of R_1 .

Theorem 4.36. *Suppose $\lambda \in \Lambda_r^+(n-2)$. Let $r_{\lambda, p, \tilde{p}} = \dim \Delta(0, \lambda \cup p \cup \tilde{p})$ if $\lambda \cup p \cup \tilde{p}$ is an multipartition. If $2 \nmid r$ and $q^{-1} = \varepsilon \prod_{i=1}^r u_i$, we define*

$$B = \prod_{\lambda \cup p \cup p^+ \in \Lambda_r^+(n)} (c_\lambda(p) - \varepsilon q^{-1})^{r_{\lambda, p, p^+}} \prod_{\lambda \cup p \cup p^- \in \Lambda_r^+(n)} (c_\lambda(p) + \varepsilon q)^{r_{\lambda, p, p^-}}.$$

Otherwise, we define

$$B = \begin{cases} \prod_{\lambda \cup p \cup p^- \in \Lambda_r^+(n)} (c_\lambda(p)^2 - q^2)^{r_{\lambda, p, p^-}}, & \text{if } 2 \mid r, q^{-1} = q^{-1} \prod_{i=1}^r u_i, \\ \prod_{\lambda \cup p \cup p^+ \in \Lambda_r^+(n)} (c_\lambda(p)^2 - q^{-2})^{r_{\lambda, p, p^+}}, & \text{if } 2 \mid r, q^{-1} = -q \prod_{i=1}^r u_i. \end{cases}$$

Then there is an $A \in R_1$ such that

$$(4.37) \quad \det G_{1, \lambda} = AB \prod_{p, \tilde{p} \in \mathcal{A}(\lambda)} (c_\lambda(p) c_\lambda(\tilde{p}) - 1)^{\dim \Delta(0, \lambda \cup p \cup \tilde{p})}.$$

Proof. Suppose that there are s (resp. $m-s$) addable (resp. removable) nodes p_1, p_2, \dots, p_s (resp. $p_{s+1}, p_{s+2}, \dots, p_m$) in $Y(\lambda)$. Let

$$\mu[i] = \begin{cases} \lambda \cup p_i, & \text{if } 1 \leq i \leq s, \\ \lambda/p_i, & \text{if } s+1 \leq i \leq m. \end{cases}$$

We need (4.38)–(4.39) which can be verified directly. Suppose $s+1 \leq k \leq m$.

$$(4.38) \quad \begin{aligned} & \{(p, \tilde{p}) \mid p, \tilde{p} \in \mathcal{A}(\mu[k]), p \neq \tilde{p}\} \\ &= \{(p, \tilde{p}) \mid p, \tilde{p} \in \mathcal{A}(\mu[k]) \cap \mathcal{A}(\lambda), p \neq \tilde{p}\} \cup \{(p, p_k) \mid p \in \mathcal{A}(\mu[k])\} \end{aligned}$$

and

$$(4.39) \quad \begin{aligned} & \{(p_i, p_k) \mid s+1 \leq k \leq m, 1 \leq i \leq s\} \\ &= \cup_{k=s+1}^m \{(p, p_k) \mid p \in \mathcal{A}(\mu[k])\} \cup \cup_{k=s+1}^m \{(p_k, p_k^+), (p_k, p_k^-)\}. \end{aligned}$$

Now, we prove the result by induction on n . It is routine to check (4.37) for the case $n=2$. Suppose $n \geq 3$. By Theorem 4.31,

$$(4.40) \quad \det G_{1, \lambda} = \prod_{i=1}^s \det G_{0, \mu[i]} \cdot \gamma_{\lambda/\mu[i]}^{\dim \Delta(0, \mu[i])} \prod_{j=s+1}^m \det G_{1, \mu[j]} \cdot \gamma_{\lambda/\mu[j]}^{\dim \Delta(1, \mu[j])}$$

By Proposition 4.15, $\det G_{0, \mu[i]} \in R_1$ and $\gamma_{\lambda/\mu[i]} \in F_1$ for $1 \leq i \leq s$ and $s+1 \leq j \leq m$. Suppose $1 \leq i \leq s$. By Propositions 4.18, 4.24,

$$(4.41) \quad \gamma_{\lambda/\mu[i]} = CD \frac{\prod_{1 \leq j \neq i \leq s} (c_\lambda(p_i) c_\lambda(p_j) - 1)}{\prod_{s+1 \leq k \leq m} (c_\lambda(p_i) - c_\lambda(p_k))}$$

where $C \in F_1$ and

$$D = \begin{cases} (c_\lambda(p_i) + \varepsilon q)(c_\lambda(p_i) - \varepsilon q^{-1}), & \text{if } 2 \nmid r, q^{-1} = \varepsilon \prod_{i=1}^r u_i, \\ c_\lambda(p_i)^2 - q^{2\varepsilon}, & \text{if } 2 \mid r, q^{-1} = \varepsilon q^{-\varepsilon} \prod_{i=1}^r u_i. \end{cases}$$

By induction assumption, $\det G_{1, \mu[j]}$ can be computed by (4.37) if $s+1 \leq j \leq m$. We rewrite the terms on the right hand side of (4.40) so as to get $(c_\lambda(p) c_\lambda(\tilde{p}) - 1)^{r_{\lambda, p, \tilde{p}}}$

in $\det G_{1,\lambda}$. In fact, this follows from (4.38) and the classical branching rule for $\Delta(0, \lambda \cup p \cup \bar{p})$. Now, (4.37) follows from similar computation together with (4.38)-(4.39). \square

5. INDUCTION AND RESTRICTION

In this section, we consider $\mathcal{B}_{r,n}$ over a field F .

Let $\mathcal{B}_{r,n}\text{-mod}$ be the category of right $\mathcal{B}_{r,n}$ -modules. We define two functors

$$\mathcal{F}_n : \mathcal{B}_{r,n}\text{-mod} \rightarrow \mathcal{B}_{r,n-2}\text{-mod}, \text{ and } \mathcal{G}_{n-2} : \mathcal{B}_{r,n-2}\text{-mod} \rightarrow \mathcal{B}_{r,n}\text{-mod}$$

such that

$$\mathcal{F}_n(M) = ME_{n-1} \text{ and } \mathcal{G}_{n-2}(N) = N_{\mathcal{B}_{r,n-2}} \otimes E_{n-1}\mathcal{B}_{r,n},$$

for all right $\mathcal{B}_{r,n}$ -modules M and right $\mathcal{B}_{r,n-2}$ -modules N . By Lemma 3.1, \mathcal{F}_n and \mathcal{G}_{n-2} are well-defined. For the simplification of notation, we will omit the subscripts of \mathcal{F}_n and \mathcal{G}_{n-2} later on.

Lemma 5.1. *Suppose that $(f, \lambda) \in \Lambda_{r,n}^+$ and $(\ell, \mu) \in \Lambda_{r,n+2}^+$.*

- a) $\mathcal{F}\mathcal{G} = 1$.
- b) $\mathcal{G}(\Delta(f, \lambda)) = \Delta(f+1, \lambda)$.
- c) $\mathcal{F}(\Delta(f, \lambda)) = \Delta(f-1, \lambda)$.
- d) *As right $\mathcal{B}_{r,n}$ -modules, $\text{Hom}_{\mathcal{B}_{r,n+2}}(E_{n+1}\mathcal{B}_{r,n+2}, \Delta(\ell, \mu)) \cong \Delta(\ell, \mu)E_{n+1}$.*
- e) *$\text{Hom}_{\mathcal{B}_{r,n+2}}(\mathcal{G}(\Delta(f, \lambda)), \Delta(\ell, \mu)) \cong \text{Hom}_{\mathcal{B}_{r,n}}(\Delta(f, \lambda), \mathcal{F}(\Delta(\ell, \mu)))$ as F -modules.*

Proof. (a) follows from Lemma 3.1, immediately. By standard arguments, we define $\psi : \Delta(f, \lambda) \otimes E_{n+1}\mathcal{B}_{r,n+2} \rightarrow \Delta(f+1, \lambda)$ such that

$$\psi((E^{f,n}M_\lambda + \mathcal{B}_{r,n}^{\triangleright(f,\lambda)}) \otimes E_{n+1}h) = E^{f+1,n+2}M_\lambda h + \mathcal{B}_{r,n+2}^{\triangleright(f+1,\lambda)}$$

for $h \in \mathcal{B}_{r,n+2}$. Since $E^{f+1,n+2}M_\lambda$ generates $\Delta(f+1, \lambda)$ as $\mathcal{B}_{r,n+2}$ -module, ψ is an epimorphism. Note that $E^{f,n} = E^{f,n}E^{f,n-1}E^{f,n}$. We have

$$\Delta(f, \lambda) \otimes E_{n+1}\mathcal{B}_{r,n+2} = (M_\lambda E^{f,n}E^{f,n-1} + \mathcal{B}_{r,n}^{\triangleright(f,\lambda)}) \otimes E^{f+1,n+2}\mathcal{B}_{r,n+2}.$$

By Lemma 2.11, $E^{f+1,n+2}\mathcal{B}_{r,n+2}$ can be written as F -linear combination of elements in $\mathcal{B}_{r,n-2f}E^{f+1,n+2}T_dX^{\kappa_d}$ where $d \in \mathcal{D}_{f+1,n+2}$ and $\kappa_d \in \mathbb{N}_r^{f+1,n+2}$. By [19, 5.8],

$$(M_\lambda E^{f,n}E^{f,n-1} + \mathcal{B}_{r,n}^{\triangleright(f,\lambda)}) \otimes \mathcal{B}_{r,n-2f}E^{f+1,n+2} = (M_\lambda E^{f,n}\mathcal{H}_{r,n-2f} + \mathcal{B}_{r,n}^{\triangleright(f,\lambda)}) \otimes E_{n+1}.$$

Therefore, $\dim_F(\Delta(f, \lambda) \otimes E_{n+1}\mathcal{B}_{r,n+2}) \leq \dim_F \Delta(f+1, \lambda)$. So, ψ is injective. This completes the proof of (b). (c) follows from (a)-(b), immediately.

We define the F -linear map $\phi : \text{Hom}_{\mathcal{B}_{r,n+2}}(E_{n+1}\mathcal{B}_{r,n+2}, \Delta(\ell, \mu)) \rightarrow \Delta(\ell, \mu)E_{n+1}$ such that $\phi(f) = f(E_{n+1})$, for $f \in \text{Hom}_{\mathcal{B}_{r,n+2}}(E_{n+1}\mathcal{B}_{r,n+2}, \Delta(\ell, \mu))$. Note that $f(E_{n+1}) \in \Delta(\ell, \mu)E_{n+1}$. So, ϕ is an epimorphism. Note that any $f \in \text{Hom}_{\mathcal{B}_{r,n+2}}(E_{n+1}\mathcal{B}_{r,n+2}, \Delta(\ell, \mu))$ is determined uniquely by $f(E_{n+1})$. So, ϕ is injective. This proves (d). Finally, (e) follows from adjoint associativity and (d). \square

Given two $\mathcal{B}_{r,n}$ -modules M, N . Let $\langle M, N \rangle_n = \dim_F \text{Hom}_{\mathcal{B}_{r,n}}(M, N)$. By Lemma 5.1(e), we have the following result immediately.

Theorem 5.2. *Given $(f, \lambda) \in \Lambda_{r,n+2}^+$ and $(\ell, \mu) \in \Lambda_{r,n+2}^+$ with $f \geq 1$. Then $\langle \Delta(f, \lambda), \Delta(\ell, \mu) \rangle_{n+2} = \langle \Delta(f-1, \lambda), \Delta(\ell-1, \mu) \rangle_n$.*

6. A CRITERION ON $\mathcal{B}_{r,n}$ BEING SEMISIMPLE

In this section, we consider $\mathcal{B}_{r,n}$ over a field F . The main purpose of this section is to give a necessary and sufficient condition for $\mathcal{B}_{r,n}$ being semisimple over F .

In Propositions 6.1–6.5, we assume $o(q^2) > n$ and $|d| \geq n$ whenever $u_i u_j^{-1} - q^{2d} = 0$ and $d \in \mathbb{Z}$. So, $\mathcal{H}_{r,n}$ is semisimple over F [1]. By Theorem 4.36, we describe explicitly when $\det G_{1,\lambda} \neq 0$ for all $\lambda \in \Lambda_n$ where Λ_n is defined in Definition 6.4.

Proposition 6.1. $G_{1,\emptyset} \neq 0$ if and only if the following conditions hold:

- a) $u_i u_j - 1 \neq 0$ for all $1 \leq i \neq j \leq r$,
- b) $u_i \notin \{-\varepsilon q, \varepsilon q^{-1}\}$ if $2 \nmid r$ and $\varrho^{-1} = \varepsilon \prod_{i=1}^r u_i$.
- c) $u_i \notin \{-q^\varepsilon, q^\varepsilon\}$ if $2 \mid r$ and $\varrho^{-1} = \varepsilon q^{-\varepsilon} \prod_{i=1}^r u_i$.

Proposition 6.2. Suppose that $n \geq 3$. Let $\lambda \in \Lambda_r^+(n-2)$ with $\lambda^{(m)} = (n-2)$ for some positive integer $m \leq r$. $\det G_{1,\lambda} \neq 0$ if and only if the following conditions hold:

- a) $u_m \notin \{q^{3-n}, -q^{3-n}\}$,
- b) $u_i u_m \notin \{q^{4-2n}, q^2\}$, for all $1 \leq i \leq r$ and $i \neq m$
- c) $u_i u_j \neq 1$ for all $m \notin \{i, j\}$ and $i \neq j$.
- d) $u_m \notin \{-\varepsilon q^3, \varepsilon q^{3-2n}, \varepsilon q\}$ and $u_i \notin \{-\varepsilon q, \varepsilon q^{-1}\}$ for all $i \neq m$ if $2 \nmid r$ and $\varrho^{-1} = \varepsilon \prod_{j=1}^r u_j$.
- e) $u_m \notin \{-q^3, q^3\}$ and $u_i \notin \{q, -q\}$ for all $i \neq m$ if $2 \mid r$ and $\varrho^{-1} = q^{-1} \prod_{j=1}^r u_j$.
- f) $u_m \notin \{-q^{3-2n}, q^{3-2n}, -q, q\}$ and $u_i \notin \{q^{-1}, -q^{-1}\}$ if $2 \mid r$ and $\varrho^{-1} = -q \prod_{j=1}^r u_j$.

Lemma 6.3. Suppose that $n \geq 3$. Let $\varepsilon = \pm 1$. Let $\lambda \in \Lambda_r^+(n-2)$ with $\lambda^{(m)} = (1^{n-2})$. $\det G_{1,\lambda} \neq 0$ if and only if the following conditions hold:

- a) $u_m \notin \{q^{n-3}, -q^{n-3}\}$,
- b) $u_i u_m \notin \{q^{2n-4}, q^{-2}\}$, for all $1 \leq i \leq r$ and $i \neq m$
- c) $u_i u_j \neq 1$ for all $m \notin \{i, j\}$ and $i \neq j$.
- d) $u_m \notin \{\varepsilon q^{-3}, -\varepsilon q^{2n-3}, -\varepsilon q^{-1}\}$ and $u_i \notin \{-\varepsilon q, \varepsilon q^{-1}\}$ for all $i \neq m$ if $2 \nmid r$ and $\varrho^{-1} = \varepsilon \prod_{j=1}^r u_j$.
- e) $u_m \notin \{-q^{2n-3}, q^{2n-3} - q^{-1}, q^{-1}\}$ and $u_i \notin \{q, -q\}$ for all $i \neq m$ if $2 \mid r$ and $\varrho^{-1} = q^{-1} \prod_{j=1}^r u_j$.
- f) $u_m \notin \{-q^{-3}, q^{-3}\}$ and $u_i \notin \{q^{-1}, -q^{-1}\}$ if $2 \mid r$ and $\varrho^{-1} = -q \prod_{j=1}^r u_j$.

Definition 6.4. Fix positive integers r and n . let

$$\Lambda_n = \bigcup_{k=2}^n \{\lambda \in \Lambda_r^+(k-2) \mid \lambda^{(i)} \in \{(k-2), (1^{k-2})\} \text{ for some } i, 1 \leq i \leq r\}$$

Proposition 6.5. Suppose that $r \geq 2$ and $n \geq 2$.

- a) Assume $\det G_{1,\emptyset} \neq 0$. Then $\prod_{\lambda \in \Lambda_n} \det G_{1,\lambda} \neq 0$ if and only if $\mathcal{B}_{r,n}$ is (split) semisimple over F .
- b) $\mathcal{B}_{r,n}$ is not semisimple over F if $\det G_{1,\emptyset} = 0$.

Proof. By Propositions 6.1–6.3, $\prod_{\lambda \in \Lambda_n \setminus \Lambda_{n-1}} \det G_{1,\lambda} = 0$ if $\det G_{1,\emptyset} = 0$. This proves (b).

We are going to prove (a) by induction on n . When $n = 2$, there is nothing to be proved. We assume $n \geq 3$ in the remainder of the proof.

In [11], Graham and Lehrer proved that a cellular algebra is (split) semisimple if and only if no Gram determinant associated to a cell module which is defined by a cellular basis is equal to zero. We use it frequently in the proof of this proposition.

(\Rightarrow) If $\mathcal{B}_{r,n}$ is not semisimple, then $\det G_{f,\lambda} = 0$ for some $(f, \lambda) \in \Lambda_{r,n}^+$. Under our assumption, $\mathcal{H}_{r,n}$ is semisimple. Since each cell module $\Delta(0, \lambda)$ for $\mathcal{B}_{r,n}$ can be considered as the cell module of $\mathcal{H}_{r,n}$ with respect to λ . So, $\det G_{0,\lambda} \neq 0$ for all $\lambda \in \Lambda_r^+(n)$. Therefore, we can assume that $f > 1$.

Take an irreducible module $D^{\ell,\mu} \subset \text{Rad } \Delta(f, \lambda)$. By general theory about cellular algebras, we know that $\ell \leq f$. When $\ell > 1$, we use Theorem 5.2 to get a non-zero $\mathcal{B}_{r,n-2}$ -homomorphism from $\Delta(\ell-1, \mu)$ to $\Delta(f-1, \lambda)$. So, $\mathcal{B}_{r,n-2}$ is not semisimple. This contradicts to our assumption since $\Lambda_{n-2} \subset \Lambda_n$. If $\ell = 0$, then there is a non-zero homomorphism from $\text{Ind}_{\mathcal{B}_{r,n-1}} \Delta(0, \mu/p)$ to $\Delta(f, \lambda)$ where p is a removable node of μ and μ/p is obtained from μ by removing the removable node p . Here we use classical branching rule for $\Delta(0, \mu/p)$ since we are assuming that $\mathcal{H}_{r,n}$ is semisimple. By Theorem 3.11, there is a $(k, \alpha) \in \Lambda_{r,n-1}$ with $(k, \alpha) \rightarrow (f, \lambda)$ such that $\Delta(0, \mu/p)$ is a composition factor of $\Delta(k, \alpha)$. Since we are assuming that $f > 1$, $k \geq f-1 > 0$. So, $(0, \mu/p) \neq (k, \alpha)$. Therefore, $\mathcal{B}_{r,n-1}$ is not semisimple. This contradicts our induction assumption again.

(\Leftarrow) By assumption, $\det G_{1,\lambda} \neq 0$ for all $\lambda \in \Lambda_n \setminus \Lambda_{n-1}$. Suppose that $\det G_{1,\lambda} = 0$ for $\lambda \in \Lambda_{n-1}$. We can find an irreducible module $D^{\ell,\mu} \subset \text{Rad } \Delta(1, \lambda)$. We have $\ell = 0$. Otherwise, since $\ell \leq 1$, we have $\ell = 1$. By Theorem 5.2, $\lambda = \mu$, a contradiction.

If $n-2-|\lambda| = 2a$ for some $a \in \mathbb{N}$, we can use Theorem 5.2 to get a non-zero homomorphism from $\Delta(a, \mu)$ to $\Delta(1+a, \lambda)$. So, $\det G_{1+a,\lambda} = 0$, forcing $\mathcal{B}_{r,n}$ not being semisimple, a contradiction.

Suppose $n-2-|\lambda|$ is odd. By Theorem 4.36, we can find a suitable multipartition, say $\tilde{\lambda}$ which is obtained from λ by adding an addable node, such that $\det G_{1,\tilde{\lambda}} = 0$. First, we assume that $\lambda \in \Lambda_r^+(k-2)$ with $\lambda^{(m)} = k-2$ and $k \leq n-1$ without loss of generality. By Proposition 6.2, either $u_i \in \{q^a, -q^b\}$ or $u_i u_j = q^c$ for some $1 \leq i \neq j \leq r$ and some integers a, b, c . In the first case, we add a box on $\lambda^{(j)}$ with $j \neq i$. In the remainder case, we define $\tilde{\lambda}^{(m)} = (k-2, 1)$ (resp. $\tilde{\lambda}^{(m)} = (k-1)$) if $u_i u_m = q^{4-2k}$ (resp. otherwise). In each case, $\tilde{\lambda} \in \Lambda_r^+(k-1)$ and $\det G_{1,\tilde{\lambda}} = 0$. Since $n-2-|\tilde{\lambda}|$ is a non-negative even number, we get a contradiction by our previous arguments.

By similar arguments, we get a contradiction if we assume $\lambda \in \Lambda_r^+(k-2)$. We leave the details to the reader. \square

For convenience, we define

$$(6.6) \quad Q_{r,\varrho} = \begin{cases} \{-\varepsilon q, \varepsilon q^{-1}\}, & \text{if } 2 \nmid r, \varrho^{-1} = \varepsilon \prod_{i=1}^r u_i, \\ \{-q^\varepsilon, q^\varepsilon\}, & \text{if } 2 \mid r, \varrho^{-1} = \varepsilon q^{-\varepsilon} \prod_{i=1}^r u_i, \end{cases}$$

and

$$(6.7) \quad S_{r,\varrho} = \begin{cases} \cup_{k=3}^n \{\pm q^{3-k}, \pm q^{k-3}, \varepsilon q^{3-2k}, -\varepsilon q^{2k-3}\}, & \text{if } 2 \nmid r, \varrho^{-1} = \varepsilon \prod_{i=1}^r u_i, \\ \cup_{k=3}^n \{\pm q^{3-k}, \pm q^{k-3}, \pm q^{(2k-3)\varepsilon}\}, & \text{if } 2 \mid r, \varrho^{-1} = \varepsilon q^{-\varepsilon} \prod_{i=1}^r u_i. \end{cases}$$

Theorem 6.8. *Let $n \geq 2$ and $r \geq 2$. Let $\mathcal{B}_{r,n}$ be defined over the field F which contains non-zero $u_i, 1 \leq i \leq r$, $q, q - q^{-1}$ such that the assumption 2.2 holds.*

- If either $u_i - u_j^{-1} = 0$ for different positive integers $i, j \leq r$ or $u_i \in Q_{r,\varrho}$ for some positive integer $i \leq r$, then $\mathcal{B}_{r,n}$ is not semisimple.*
- Assume $u_i - u_j^{-1} \neq 0$ for all different positive integers $i, j \leq r$ and $u_i \notin Q_{r,\varrho}$ for all positive integers $i \leq r$.*

- (1) $\mathcal{B}_{r,2}$ is semisimple if and only if $o(q^2) > 2$ and $|d| \geq 2$ whenever $u_i u_j^{-1} = q^{2d}$ for any $1 \leq i < j \leq r$ and $d \in \mathbb{Z}$.
- (2) Suppose $n \geq 3$. Then $\mathcal{B}_{r,n}$ is semisimple if and only if
 - (a) $o(q^2) > n$,
 - (b) $|d| \geq n$ whenever $u_i u_j^{-1} = q^{2d}$ for any $1 \leq i < j \leq r$ and $d \in \mathbb{Z}$,
 - (c) $u_i \notin S_{r,q}$,
 - (d) $u_i u_j \notin \cup_{k=3}^n \{q^{4-2k}, q^{2k-4}\}$ for all different positive integers $i, j \leq r$.

Proof. Each cell module $\Delta(0, \lambda)$ for $\lambda \in \Lambda_r^+(n)$ can be considered as the cell module of $\mathcal{H}_{r,n}$. So, $\mathcal{B}_{r,n}$ is not semisimple over F if $\mathcal{H}_{r,n}$ is not semisimple. Therefore, we can assume $\mathcal{H}_{r,n}$ is semisimple when we discuss the semisimplicity of $\mathcal{B}_{r,n}$. Now, the result follows from Ariki's result on $\mathcal{H}_{r,n}$ being semisimple in [1] together with Propositions 6.1-6.5. \square

When $r = 1$, Theorem 6.8 has been proved in [17, 5.9]. We remark that the notation r (resp. ω) in [17, 1.1] is the same as ρ^{-1} (resp. δ) in the current paper.

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